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MINIFOLDS AND PHANTOMS

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Abstract. A minifold is a smooth projective n-dimensional variety such that its bounded derived category of coherent sheaves admits a semi-orthogonal decomposition into a collection of $n+1$ exceptional objects. In this paper we classify minifolds of dimension $n \leqslant 4$.

We conjecture that the derived category of fake projective spaces have a similar semi-orthogonal decomposition into a collection of $n+1$ exceptional objects and a category with vanishing Hochschild homology. We prove this for fake projective planes with non-abelian automorphism group (such as Keum's surface). Then by passing to equivariant categories we construct new examples of phantom categories with both Hochschild homology and Grothendieck group vanishing.

1. Introduction.

The question of homological characterization of projective spaces goes back to Severi, and the pioneering work of Hirzebruch–Kodaira [28]. Beautiful results have been obtained by Kobayashi–Ochiai [41], Yau [62], Fujita [24], Libgober–Wood [49].

Among smooth projective varieties of given dimension projective spaces have the smallest cohomology groups. We call a smooth projective variety a $\mathbb{Q}$-*homology projective space* if it has the same Hodge numbers as a projective space. Any odd-dimensional quadric is an example of $\mathbb{Q}$-homology projective space. We call an n-dimensional $\mathbb{Q}$-homology projective space X of general type a *fake projective space* if in addition it has the same "Hilbert polynomial" as $\mathbb{P}^n$: $\chi(X, \omega_X^{\otimes l}) = \chi(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{\otimes l})$ for all $l \in \mathbb{Z}$. Any *fake projective plane* is simply a $\mathbb{Q}$-homology plane of general type, since Hodge numbers of a surface determine its Hilbert polynomial. On the level of realizations over $\mathbb{C}$, e.g. from the point of view of the Hodge structure, fake projective spaces are identical to projective spaces, however the study of their K-theory, motive or derived category meets cohomological subtleties.

The first example of a fake projective plane was constructed by Mumford [52] using p-adic uniformization developed by Drinfeld [21] and Mustafin [53]. From the point of view of complex geometry fake projective planes have been studied by Aubin [3] and Yau [62], who proved that any such surface S is uniformized by a complex ball, hence by Mostow's rigidity theorem S is determined by its fundamental group $\pi_1(S)$ uniquely up to complex conjugation; Kharlamov–Kulikov [39] shown that the conjugate surfaces are distinct (not biholomorphic). Further Klingler

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[40] and Yeung [63] proved that $\pi_1(S)$ is a torsion-free cocompact *arithmetic* subgroup of $PU(2,1)$. Finally such groups have been classified by Cartwright–Steger [18] and Prasad–Yeung [57]: there are 50 explicit subgroups and so all fake projective planes fit into 100 isomorphism classes.

Fake projective fourspaces were introduced and studied by Prasad and Yeung in [58].

In this paper we take a different perspective that started with a seminal discovery of *full exceptional collections* by Beilinson [6], Kapranov [32], Bondal and Orlov [15] with Bondal's students Kuznetsov, Razin, Samokhin (see [45]): they found out that all known to them examples of Fano $\mathbb{Q}$-homology projective spaces admit a full exceptional collection of vector bundles. They put a conjecture that gives a homological characterization of projective spaces based on derived categories, and in this paper we prove it in Theorem 1.1(3).

We call an n-dimensional smooth complex projective variety a ***minifold*** if it has a full exceptional collection of minimal possible length $n+1$ in its bounded derived category of coherent sheaves. A minifold is necessarily a $\mathbb{Q}$-homology projective space. Projective spaces and odd-dimensional quadrics are examples of minifolds [6, 32].

It follows from work of Bondal, Bondal–Polishchuk and Positselski [12, 16, 56], that if a minifold X is not Fano then all full exceptional collections on it are not strict and consist not of pure sheaves. In fact it is expected that all minifolds are Fano.

The novelty of this paper is the following main theorem, which gives a classification of minifolds in dimension less than or equal to 4 (with one-dimensional case being trivial).

Theorem 1.1. *1) The only two-dimensional minifold is* $\mathbb{P}^2$.

2) The minifolds of dimension 3 *are: the projective space* $\mathbb{P}^3$ *the quadric* Q^3, *the del Pezzo quintic threefold* V_5, *and a six-dimensional family of Fano threefolds* V_{22}.

3) The only four-dimensional Fano minifold is $\mathbb{P}^4$.

In Section 2 we recall the necessary definitions and facts. In particular in Proposition 2.1 we recall that varieties admitting full exceptional collections have Tate motives with rational coefficients [51] and outline a straightforward proof of that fact. Section 2 finishes with the proof of Theorem 1.1.

We also show that except for $\mathbb{P}^4$ the only possible minifolds of dimension 4 are non-arithmetic fake projective fourfolds, which presumably do not exist [65] (paragraph 4 and section 8.4).

In fact, study of minifolds is closely related to study of the fake projective spaces. The reason is that fake projective spaces sometimes admit exceptional collections of the appropriate length but these collections fail to be full. In this case the orthogonal to such a collection is a so-called phantom.

More precisely, we call an admissible non-zero subcategory $\mathcal{A}$ of a derived category of coherent sheaves an H**-phantom** if $HH_*(\mathcal{A})=0$ and a K**-phantom** if $K_0(\mathcal{A})=0$.

In Section 3 we formulate a conjecture that under some mild conditions fake projective n-spaces admit non-full exceptional collections of length $n+1$ and thus have H-phantoms in their derived categories (Conjecture 3.1 and its Corollary 3.2). We prove this conjecture for fake projective planes admitting an action of the non-abelian group G_{21} of order 21.

Theorem 1.2. *Let* S *be one of the six fake projective planes with automorphism group of order* 21. *Then* $K_S=\mathcal{O}(3)$ *for a unique line bundle* $\mathcal{O}(1)$ *on* S. *Furthermore* $\mathcal{O}$, $\mathcal{O}(-1)$, $\mathcal{O}(-2)$ *is an exceptional collection on* S.

Most of Section 3 deals with the proof of this Theorem, which relies on the holomorphic Lefschetz fixed point formula applied to the three fixed points of an element of order 7 as in [38].

It follows from Theorem 1.2 that $\mathcal{D}^b(S)$ has an H-phantom subcategory $\mathcal{A}_S$. We show that this H-phantom descends to an H-phantom $\mathcal{A}_S^G$ in the equivariant derived categories $\mathcal{D}_G^b(S)$ for any $G \subset G_{21}$. In particular, when $G = \mathbb{Z}/3$ this gives yet new examples of surfaces having an H-phantom in their derived categories (fake cubic surfaces: see Remark 3.8).

Finally, in four cases $G \supset \mathbb{Z}/7$ *and* surface S/G is simply-connected, then we show that the H-phantom $\mathcal{A}_S^G$ is also a K-phantom (Proposition 3.10).

In the Appendix we give a table of arithmetic subgroups $\Pi \subset PSU(2,1)$ giving rise to fake projective planes and the corresponding automorphism and first homology groups. These results are taken from the computations of Steger and Cartwright [19].

It took a long way for the paper to take its present form. We would like to thank our friends and colleagues with whom we had fruitful discussions on the topic.

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While this paper was in preparation, Najmuddin Fakhruddin gave a proof of Conjecture 3.1 for those fake projective planes that admit 2-adic uniformization [23], in particular for Mumford's fake projective plane. These cases are disjoint from the ones that satisfy the assumptions of our Theorem 1.2. An idea of a proof in the case of Mumford's fake projective plane similar to that of [23] was also hinted to us by Dolgachev. Independently it was sketched to us by van Straten in June 2013: van Straten and Spandau has an unpublished work circa 2001, where based on description of Ishida [29] they construct the reduction modulo two of the 2-canonical image of Mumford's fake projective plane as an image of $\mathbb{P}^2(\mathbb{F}_2)$ by an explicit 10-dimensional linear system of octics (plane curves of degree 8), one then checks that these octics are not divisible by 3 as Weil divisors. Our proof of Theorem 1.2 is indebted to discussions with Panov in June 2012 (he suggested us to exploit extra symmetries of Keum's surface) and with Dolgachev in June 2013 (then we looked for higher-dimensional irreducible representations of non-abelian groups). The rest of the ideas of the paper is from 2011.

2. MINIFOLDS

An *exceptional collection* of length r on a smooth projective variety $X/\mathbb{C}$ is a sequence of objects $E_1, \dots E_r$ in the bounded derived category of coherent sheaves $\mathcal{D}^b(X)$ such that $\mathrm{Hom}(E_j, E_i[k]) = 0$ for all $j > i$ and $k \in \mathbb{Z}$, and moreover each object E_i is exceptional, that is spaces $\mathrm{Hom}(E_i, E_i[k])$ vanish for all k except for one-dimensional spaces $\mathrm{Hom}(E_i, E_i)$. An exceptional collection is called *full* if the smallest triangulated subcategory which contains it, coincides with $\mathcal{D}^b(X)$.

Proposition 2.1. *Assume that X admits a full exceptional collection of length r. Then:*

(1) The Chow motive of X with rational coefficients is a direct sum of r Tate motives $\mathbb{L}^j$. In particular, all cohomology classes on X are algebraic. (For the definition and properties of Chow motives see [50]*.)*

(2) $H^{p,q}(X) = 0$ for $p \neq q$ and $\chi(X) = \sum h^{p,p}(X) = r$.

(3) Pic(X) *is a free abelian group of finite rank. Moreover the first Chern class map gives an isomorphism* $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$.

(4) $H_1(X,\mathbb{Z})=0$.

(5) The Grothendieck group $K_0(X)=K_0(\mathcal{D}^b(X))$ *is free of rank* r *and the bilinear Euler pairing*

$$\chi(E,F)=\sum_i(-1)^i\dim\operatorname{Hom}(E,F[i])$$

is non-degenerate and unimodular. Classes $[E_i]$ *of exceptional objects form a semi-orthonormal basis in* $K_0(X)$*. (By a semiorthonormal basis we mean a basis* $(e_i)_{i=0}^n$ *such that* $\chi(e_j,e_i)=0$, $j>i$ *and* $\chi(e_i,e_i)=1$*.)*

Proof. Most of the claims are well-known. (1) is proved in [51] using the language of noncommutative motives and in [27] using K-motives. We give a direct proof of (1) using the ideas developed in [55] (see also [7]) for the sake of completeness.

First observe that the structure sheaf of the diagonal $\mathcal{O}_\Delta$ in the derived category $D^b(X\times X)$ lies in the full triangulated subcategory generated by the objects $p_1^*F_1\otimes p_2^*F_2$. This can be deduced from the standard fact that if $E_1,\dots,E_r$ is a full exceptional collection on X, then $p_1^*E_i\otimes p_2^*E_j$ forms a full exceptional collection on $X\times X$ [13](see Lemma 3.4.1 and note that for the category generated by an exceptional collection the notions of generator and strong generator coincide, so taking direct summands is not necessary), [9], [59].

It follows that the class of the diagonal $[\mathcal{O}_\Delta]\in K_0(X\times X)$ has a decomposition

$$[\mathcal{O}_\Delta]=\sum_j p_1^*[\mathcal{F}_j]\cdot p_2^*[\mathcal{G}_j], \tag{2.1}$$

for some sheaves $\mathcal{F}_j,\mathcal{G}_j$ on X.

Applying the Chern character to (2.1) and using the Grothendieck-Riemann-Roch formula

$$ch(\mathcal{O}_\Delta)=[\Delta]\cdot p_2^*td(X)$$

we obtain an analogous decomposition for the class of the diagonal $[\Delta]\in CH^*(X\times X)_{\mathbb{Q}}$:

$$[\Delta]=\sum_j p_1^*\alpha_j\cdot p_2^*\beta^j, \tag{2.2}$$

for some classes $\alpha_j,\beta_j\in CH^*(X)_{\mathbb{Q}}$. We may assume that α_j are homogenous, say $\alpha_j\in CH^{a_j}(X)_{\mathbb{Q}}$ and hence $\beta_j\in CH^{dim(X)-a_j}(X)_{\mathbb{Q}}$.

We claim that the set $\{\alpha_j\}$ spans $CH^*(X)_{\mathbb{Q}}$. Indeed for any $\alpha\in CH^*(X)_{\mathbb{Q}}$ we have

$$\begin{aligned}\alpha&=p_{1*}([\Delta]\cdot p_2^*\alpha)=\\&=p_{1*}((\sum_j p_1^*\alpha_j\cdot p_2^*\beta^j)\cdot p_2^*\alpha)=\\&=p_{1*}(\sum_j p_1^*\alpha_j\cdot p_2^*(\beta^j\cdot\alpha))=\\&=\sum_j\alpha_j\cdot p_{1*}(p_2^*(\beta^j\cdot\alpha))=\\&=\sum_j\langle\beta^j,\alpha\rangle\alpha_j.\end{aligned} \tag{2.3}$$

Here we use the notation $\langle\alpha,\beta\rangle$ for the bilinear form $deg(\alpha\cdot\beta)$.

We may assume that $\{\alpha_j\}$ are linearly independent, that is form a homogeneous basis of $CH^*(X)_{\mathbb{Q}}$. From the formula (2.3) we see that $\{\beta_j\}$ is a dual basis.

We now define an isomorphism $M(X) \cong \oplus_j \mathbb{L}^a_j$. By definition of the morphisms in the category of Chow motives we have

$$Hom(\mathbb{L}^a, M(X)) = CH^a(M)$$

$$Hom(M(X), \mathbb{L}^a) = CH^{dim(X)-a}(M)$$

Therefore the set $\{\alpha_j\}$ determines a morphism of motives

$$\Phi : \oplus_j \mathbb{L}^{a_j} \to M(X)$$

and the $\{\beta_j\}$ determines a morphism in the opposite direction

$$\Psi : M(X) \to \oplus_j \mathbb{L}^{a_j}.$$

The composition $\Psi \circ \Phi$ is equal to identity due to the fact that $\{\alpha_j\}$ and $\{\beta_j\}$ are dual bases. The composition $\Phi \circ \Psi$ is equal to identity because of the decomposition (2.2).

By taking Hodge realization (1) implies (2). Alternatively, we can deduce (2) from Hochschild–Kostant–Rosenberg theorem

$$HH_i(\mathcal{D}^b(X)) \cong \oplus_{p-q=i} H^{p,q}(X)$$

and additivity of Hochschild homology for semiorthogonal decompositions [37, 46]: if $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ then $HH_i(\mathcal{C}) = HH_i(\mathcal{A}) \oplus HH_i(\mathcal{B})$.

The fact that $\mathrm{Pic}(X)$ is free follows from (5) and Lemma 2.2 below. The isomorphism $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ comes from the exponential long exact sequence and (2). $\mathrm{Pic}(X)$ is of finite rank since it is isomorphic to $H^2(X, \mathbb{Z})$.

To prove (4) note that the Universal Coefficient Theorem implies that we have a non-canonical isomorphism

$$H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\mathrm{rk}} \oplus H_1(X, \mathbb{Z})^{tors}$$

which by (3) implies that $H_1(X, \mathbb{Z})$ must be torsion-free as well. On the other hand $h^{1,0}(X) = 0$ and hence $H_1(X, \mathbb{Z}) = 0$.

(5) follows easily from definitions.

□

Lemma 2.2. *Let X be a smooth algebraic variety such that $K_0(X)$ has no p-torsion. Then* $\mathrm{Pic}(X)$ *has no p-torsion.*

Proof. We prove that if $\mathrm{Pic}(X)$ has p-torsion, then the same is true for $K_0(X)$.

Let L be a line bundle on X such that $L^{\otimes p} \cong \mathcal{O}_X$. Let $N = [L] - 1 \in K_0(X)$; then N is nilpotent. Indeed N being of rank zero, sits in the first term $F^1 K_0(X)$ of the topological filtration on $K_0(X)$ ([25], Example 15.1.5). The topological filtration is multiplicative; therefore $N^{dim(X)+1} \in F^{dim(X)+1} K_0(X) = 0$.

Let k be the smallest positive integer such that $N^k = 0$. If $k = 1$, that is $N = 0$ and $[L] = 1 \in K_0(X)$, then $L \cong \mathcal{O}_X$ since to $F^1 X / F^2 X \cong \mathrm{Pic}(X)$ by [25], Example 15.3.6.

We assume now that $k \geqslant 2$. We have

$$[L] \ = \ 1 + N.$$

Taking p-th tensor power of both sides we obtain

$$1 \ = \ 1 + pN + N^2 \alpha, \ \alpha \in K_0(X)$$

$$0 \ = \ pN + N^2 \alpha$$

and after multiplying by N^{k-2}:

$$pN^{k-1} = 0,$$

so that N^{k-1} is nontrivial p-torsion class in $K_0(X)$. $\square$

Remark 2.3. In fact if we assume X to be a compact Kähler manifold with a full exceptional collection in the analytic derived category $\mathcal{D}^b_{an}(X)$ of complexes of $\mathcal{O}_X$-modules with bounded coherent cohomology, one can show that $H^{2,0}(X,\mathbb{C}) = 0$, so that the Kähler cone is open in $H^2(X,\mathbb{R}) = H^{1,1}(X,\mathbb{C}) \cap \overline{H^{1,1}(X,\mathbb{C})}$ hence it has non-trivial intersection with $H^2(X,\mathbb{Z})$.

Then the Kodaira embedding theorem implies that X is projective.

Definition 2.4. We call a smooth projective complex variety of dimension n admitting a full exceptional collection of length $n+1$ a *minifold*.

It follows from Proposition 2.1 (2), that $n+1$ is the minimal number of objects in such a collection, and the term "minifold" originates from here. Minifolds have the same Hodge numbers as projective spaces. By results of Beilinson [6] and Kapranov [32], projective spaces and odd-dimensional quadrics are minifolds.

Lemma 2.5. *Let X be a minifold. Then*

- *either X is a Fano variety i.e. the anticanonical line bundle $\omega_X^\vee = \det T_X$ is ample*
- *or canonical line bundle $\omega_X = \det T_X^*$ is ample*

In particular, the variety X is uniquely determined by $\mathcal{D}^b(X)$.

Proof. We first note that ω_X is not trivial, since $h^0(\omega_X) = h^{n,0}(X) = 0$ by Proposition 2.1(2). By Proposition 2.1(3), $\mathrm{Pic}(X)$ is torsion free; hence the class of ω_X in $\mathrm{Pic}(X)_\mathbb{Q} \cong H^2(X,\mathbb{Q}) = \mathbb{Q}$ is non-zero. Therefore either ω_X or $\omega_X^\vee$ is ample. Now the Bondal-Orlov [15] reconstruction theorem implies the last statement. $\square$

Remark 2.6. If we weaken the assumption from "projective" to "proper" in the definition of a minifold, we still get the same class of varieties. Indeed, if X is a proper smooth variety of dimension n with a full exceptional collection of length $n+1$ we can still deduce that ω_X or its dual is ample, in particular that X is projective as follows.

From [49](Theorem 3) it follows that for a compact complex n-dimensional manifold, the Chern number c_1c_{n-1} is determined by Hirzebruch χ-genera χ_y and hence by the Hodge numbers.

Thus we have $c_1c_{n-1}[X] = c_1c_{n-1}[\mathbb{P}^n] = \frac{n(n+1)^2}{2} \neq 0$. Since the Kleiman-Mori cone of effective one-cycles modulo numerical equivalence $N_1(X) \subset H^2(X,\mathbb{R})$ is one dimensional (that is because $H^2(X,\mathbb{R})$ itself is one dimensional by Proposition 2.1(2) which still holds under the assumption that X is proper), Kleiman's criterion for ampleness implies that either ω_X or its dual is ample.

The rest of this section is devoted to proof of Theorem 1.1. In view of Lemma 2.5, the proof consists of classifying Fano minifolds and showing that there is no minifolds among varieties of general type.

We start in dimension 2. The only del Pezzo surface with Picard number one is a projective plane.

On the other hand it is known that fake projective planes have non-vanishing torsion first homology group [57], Theorem 10.1. Hence by Proposition 2.1(4) there is no minifold of general type of dimension 2.

Let us consider Fano threefolds. By Proposition 2.1(2) conditions $b_2(X) = 1$ and $b_3(X) = 0$ are necessary for a minifold. Such Fano threefolds were classified by Iskovskikh [31] into four deformation types: the projective space $\mathbb{P}^3$, the quadric Q^3, the del Pezzo quintic threefold V_5, and a family of Fano threefolds V_{22}.

All these varieties are known to admit an exceptional collection of length 4 by results of Beilinson, Kapranov, Orlov and Kuznetsov respectively [6], [32], [54], [44].

It is easy to see that 3-dimensional $\mathbb{Q}$-homology varieties of general type do not exist. Indeed K_X ample implies that $c_1(X)^3$ is negative, but by Todd's theorem $c_1(X)c_2(X) = 24$. This contradicts to Yau's inequality $c_1(X)^3 \geqslant \frac{8}{3}c_2(X)c_1(X)$ [62].

According to Wilson [61] and Yeung [64] there are three alternatives for a $\mathbb{Q}$-homology projective fourspace X: either X is $\mathbb{P}^4$, or X is a fake projective fourspace, or X has Hilbert polynomial $\chi(\omega_X^{-l}) = 1 + \frac{25}{8}l(l+1)(3l^2+3l+2)$ and Chern numbers $[c_1^4, c_2c_1^2, c_2^2, c_1c_3, c_4] = [225, 150, 100, 50, 5]$. In what follows the varieties of the latter type are named *Wilson's fourfolds.*

There are some known examples of fake projective fourfolds, but it is not known whether any Wilson's fourfold actually exist.

In what follows we show that (possibly non-existent) Wilson's fourfolds do not satisfy conditions of Proposition 2.1(5), and hence do not admit a full exceptional collection. In order to do that we relate the Grothendieck group of a minifold to its Hilbert polynomial.

We need a simple Lemma from linear algebra.

Lemma 2.7. *Let* $P(x) = \sum_{j=0}^{n} p_j x^j \in K[x]$ *be a polynomial of degree* $\leqslant n$ *with coefficients in a field* K *of characteristic zero and let* A_P *be the* $(n+1) \times (n+1)$*-matrix with coefficients* $a_{i,j} = P(j-i)$. *Then we have*

$$det(A_P) = (n!\, p_n)^{n+1}.$$

In particular the matrix A_P *is non-degenerate if and only if* $\deg P = n$.

Proof. It suffices to prove the statement for algberaic closure $\bar{K}$ of K, we thus assume K to be algebraically closed.

We first prove that

$$det(A_P) = 0 \Longleftrightarrow p_n = 0. \tag{2.4}$$

Indeed if $deg(P(x)) < n$, then $n+1$ polynomials $P(x), P(x+1), \ldots, P(x+n)$ are linearly dependent which makes the columns of A_P linearly dependent, thus $det(A_P) = 0$. On the other hand, it is easy to see that if $deg(P(x)) = n$, then

$$P(x), P(x+1), \ldots, P(x+n)$$

form a basis of the space of polynomials of degree $\leqslant n$, and A_P is a matrix of an invertible linear transformation $P \mapsto (P(0), P(-1), \ldots, P(-n)) \in K^{n+1}$ in this basis, hence $det(A_P) \neq 0$.

Let $F(p_0, p_1, \ldots, p_n) = det(A_P)$. Since the entries of the matrix A_P are linear forms in $p_0, p_1, \ldots, p_n$, it follows that F is homogeneous in p_i's of degree $n+1$. Then (2.4) says that the support of the degree $n+1$ hypersurface $F = 0$ in $\mathbb{P}^n$ is contained in the hyperplane $p_n = 0$. Therefore

$$F(p_0, p_1, \ldots, p_n) = C_n \cdot p_n^{n+1} \tag{2.5}$$

for some constant $C_n \in K$. In particular $det(A_P)$ takes the same value C_n for any monic polynomial $P(x)$ of degree n.

Let $P_0(x) = (x+1) \cdot (x+2) \cdot \cdots \cdot (x+n)$. Then the matrix A_{P_0} is uppertriangular with all diagonal entries equal to $n!$:

$$C_n = det(A_{P_0}) = (n!)^{n+1} \tag{2.6}$$

The result now is the combination of (2.5) and (2.6) $\square$

Proposition 2.8. *Let X be a minifold. Let $\mathcal{O}(1) = det(T_X)$ be the anticanonical bundle,* $\deg(X)$ *be the anticanonical degree $c_1(X)^n$ and $P_X(k) = \chi(\mathcal{O}(k))$ be the Hilbert polynomial. Consider a sublattice $\Lambda \subset K_0(X)$ spanned by*

$$[\mathcal{O}], [\mathcal{O}(1)], \ldots, [\mathcal{O}(n)].$$

Then the Euler pairing restricted to Λ is non-degenerate, that is classes $[\mathcal{O}], [\mathcal{O}(1)], \ldots, [\mathcal{O}(n)]$ are linearly independent in $K_0(X)$ and Λ is a sublattice in $K_0(X)$ of full rank. Furthermore, Λ admits a semi-orthonormal basis over the ring $\mathbb{Z}[\frac{1}{deg(X)}]$ and hence modulo any prime p that does not divide $deg(X)$.

Proof. Let A_X denote the matrix of the pairing on Λ, that is a matrix with entries $a_{i,j} = \chi(\mathcal{O}(i), \mathcal{O}(j)) = P_X(j - i)$.

We apply Lemma 2.7 to $P = P_X$, the Hilbert polynomial. Its top coefficient is equal to $p_n = \frac{\deg(X)}{n!}$; therefore $det(A_X) = deg(X) \neq 0$ is the anticanonical degree and the pairing on Λ is non-degenerate.

The inclusion $\Lambda \subset K_0(X)$ becomes an isomorphism after inverting $det(A_X) = deg(X)$. Indeed let $e_j, j = 0, \ldots n$ be a basis in $K_0(X)$ and write

$$[\mathcal{O}(i)] = \sum G_{j,i} e_j, \ 0 \leqslant i \leqslant n.$$

The matrix $G^{-t} A_X G^{-1}$ is unimodular, hence $deg(X) = det(G)^2$, and after inverting $deg(X)$, G becomes invertible.

Since $K_0(X)$ admits a semiorthonormal basis by assumption and Proposition 2.1(5), the same holds for $\Lambda \otimes \mathbb{Z}[\frac{1}{deg(X)}]$ $\square$

Let P_X be the Hilbert polynomial of Wilson's fourfolds and A_X be the 5×5-matrix $(A_X)_{i,j} = P_X(j - i)$. Consider their residues modulo two: $\overline{A}_X = A_X \mod 2$, $(\overline{A}_X)_{i,j} = \overline{P}_X(j - i)$.

In the Proof of Proposition 2.8 we showed that the determinant of matrix A_X equals $deg(X)^{n+1} = 225^5 = 15^{10}$, hence the assumption that X is a minifold would imply that A_X admits a semiorthonormal basis modulo all primes $p \neq 3, 5$, in particular this would imply that $\overline{A}_X$ has a semiorthonormal basis.

Entries of A_X and $\overline{A}_X$ are determined by values $P(n)$ for $0 \leqslant n \leqslant 4$ (that we tabulate) and Serre duality $P(n) = P(-1 - n)$:

n	0	1	2	3	4
$P(n)$	1	51	376	1426	3876
$P(n) \mod 2$	1	1	0	0	0

$$\overline{A}_X = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The following Lemma gives a contradiction, from which we see that a Wilson fourfold X can not be a minifold.

Lemma 2.9. *Let $(u, v) \mapsto u^t \overline{A}_X v$ be the bilinear form on a vector space $V = \mathbb{F}_2^5$ given by the matrix $\overline{A}_X$. There is no basis e_1, e_2, e_3, e_4, e_5 of V such that $(e_i, e_j) = 0$ for $i > j$ and $(e_i, e_i) = 1$.*

Proof. We begin by making a few remarks.

1. Let $S := \overline{A}_X^{-1}\overline{A}_X^t$ be an automorphism of V. In fact S is induced by the Serre functor $\mathcal{S}_X = \otimes\omega_X[\dim X]$ on $\mathcal{D}^b(X)$ [12, 14]. S satisfies $(u, v) = (v, Su)$ for all u, v, so it preserves $\overline{A}_X$, i.e. $(u, v) = (Su, Sv)$, equivalently $S^t\overline{A}_XS = \overline{A}_X$. We have
$$S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
and S has order 8 because the value of $P(n) \mod 2$ depends only on $n \mod 8$.
2. There are precisely 12 vectors x such that $(x, x) = 1$. Indeed $(x, x) = 1$ if and only if the point x does not lie on quadric $Q = \{x|(x, x) = 0\}$. The quadric Q has a unique singular point in $\mathbb{P}(V)$ so it has 19 points over $\mathbb{F}_2$ and its complement has 12 points. These twelve points form two orbits under the action of S. One orbit of length 8 is generated by $a_1 := (1, 0, 0, 0, 0)^t$, another orbit of length 4 is generated by $b_1 := (1, 0, 1, 0, 0)^t$.
3. If a basis $e_1, e_2, \ldots, e_5$ is semi-orthonormal, then for each i $(1 \leqslant i \leqslant 4)$ the basis obtained by replacing e_i, e_{i+1} with $e_{i+1}, e_i + e_{i+1}(e_i, e_{i+1})$ is also semi-orthonormal. This transformation corresponds to mutations of exceptional collections [12, 14].

Denote $a_i = S^{i-1}a_1$, $b_i = S^{i-1}b_1$ and $c = (a_1, \ldots, a_8, b_1, \ldots, b_4)$. The following matrix has (c_i, c_j) on position i, j:

$$\left(\begin{array}{ccccccc|c|cccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}\right)$$

Assume there exists a semi-orthonormal basis. Then all of its vectors must be from the set $\{a_i\} \cup \{b_i\}$. Since there are only 4 vectors in $\{b_i\}$, at least one of the basis vectors must be from $\{a_i\}$. Applying S if necessary we may assume that this vector is a_1. Applying the transformation (3) we can obtain a semi-orthonormal basis with a_1 on the first position.

Any remaining basis vector x must satisfy $(x, a_1) = 0$. Looking at the first column of the matrix of (c_i, c_j) we see that the remaining basis vectors must be from the set $\{a_3, a_4, a_5, a_6, b_1, b_2\}$. Let x be the second basis vector. Then any vector y out of the remaining 3 basis vectors must satisfy $(y, x) = 0$. However, trying for x each of the $\{a_3, a_4, a_5, a_6, b_1, b_2\}$ we see that there are only 2 choices remaining for y. This is a contradiction. □

We also can prove that there is no minifolds among arithmetic fake projective fourspaces. This goes similarly to dimension 2 case: Prasad and Yeung proved that for an arithmetic fake projective fourspace the first homology group $H_1(X, \mathbb{Z})$ is non-zero [58], Theorem 4. Therefore by Proposition 2.1(3) these fourfolds are not minifolds.

3. PHANTOMS IN FAKE PROJECTIVE SPACES

Fake projective spaces seem to be very similar and yet very different from ordinary projective spaces. We propose the following conjecture.

Conjecture 3.1. *Assume that X is an n-dimensional fake projective space with canonical class divisible by $(n+1)$. Then for some choice of $\mathcal{O}(1)$ such that $\omega_X = \mathcal{O}(n+1)$, the sequence*

$$\mathcal{O}, \mathcal{O}(-1), \dots, \mathcal{O}(-n)$$

is an exceptional collection on X.

We call an non-zero admissible subcategory $\mathcal{A} \subset \mathcal{D}^b(X)$ an H***-phantom*** if $HH_{\bullet}(\mathcal{A}) = 0$ and a K***-phantom*** if $K_0(\mathcal{A}) = 0$.

Corollary 3.2. *Fake projective spaces as in Conjecture admit an H-phantom admissible subcategories in their derived categories $\mathcal{D}^b(X)$.*

Proof. Assume that $\mathcal{O}, \mathcal{O}(-1), \dots, \mathcal{O}(-n)$ is an exceptional collection, and consider its right orthogonal $\mathcal{A}$. By results of Bondal and Kapranov [12, 14] the category $\mathcal{A}$ is admissible, and thus we have a semi-orthogonal decomposition:

$$\mathcal{D}^b(X) = \langle \mathcal{O}, \mathcal{O}(-1), \dots, \mathcal{O}(-n), \mathcal{A} \rangle \,.$$

Note that this exceptional collection could not be full at least for two reasons:

- if $\mathcal{O}(i)$ would be a full collection then by [16](Theorem 3.4) or [56](see proof of main theorem) manifold X would be Fano, which contradicts to general type assumption,
- use Corollary 4.6 and Proposition 4.7 of [47]: by Kodaira vanishing for $i < j$ space $Ext^k(\mathcal{O}(-i), \mathcal{O}(-j))$ vanish unless $k = n$, so relative height of any two objects in a helix $\mathcal{O}(i)$ equals n, thus pseudoheight of the collection coincides with its height and is equal to $n-1$, hence Hochschild cohomology $HH^0(\mathcal{A}) = HH^0(X) \neq 0$ for $n > 1$.

Finally, Hochschild homology is additive for semi-orthogonal decompositions (cf the alternative proof of 2.1(2)), so $\dim HH_{\bullet}(\mathcal{A}) = 0$ that is $\mathcal{A}$ is an H-phantom. □

Remark 3.3. 1. A statement analogous to Conjecture 3.1 holds for some fake del Pezzo surfaces of degrees one [10, 11], six [1] and eight [26, 48]. Here we add degrees three (Remark 3.8) and nine (Theorem 1.2).

2. Fake projective planes with properties as in Conjecture 3.1 are constructed in [57], 10.4. Choose $\mathcal{O}(1)$ such that $\mathcal{O}(3) = \omega_X$. Then by the Riemann-Roch theorem the Hilbert polynomial is given by

$$\chi(\mathcal{O}(k)) = \frac{(k-1)(k-2)}{2}.$$

Therefore the collection $E_{\bullet} = (\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2))$ is at least numerically exceptional, that is

$$\chi(E_j, E_i) = 0, \; j > i.$$

In addition we have

$$H^0(S, \mathcal{O}(1)) = H^0(S, \mathcal{O}(3)) = 0.$$

Furthermore it follows from Serre duality that a necessary and sufficient condition for $E_\bullet$ to be exceptional is vanishing of the space of the global sections $H^0(S,\mathcal{O}(2))$. It is not hard to see that for all fake projective planes $h^0(S,\mathcal{O}(2)) \leqslant 2$ (cf end of the Proof of Theorem 1.2).

3. More generally our definition of an n-dimensional fake projective space includes that its Hilbert polynomial is the same as that of a $\mathbb{P}^n$. It follows that if we assume $\omega_X = \mathcal{O}(n+1)$, then we have

$$\chi(\mathcal{O}(k)) = (-1)^n \frac{(k-1)(k-2)\dots(k-n)}{n!},$$

so that $k = 1,\dots,n$ are the roots of χ, and the collection

$$\mathcal{O}, \mathcal{O}(-1), \dots, \mathcal{O}(-n)$$

is numerically exceptional.

4. G.Prasad and S.-K. Yeung informed us that the assumption $\omega_X = \mathcal{O}(5)$ is known to be true for the four arithmetic fake projective fourspaces constructed in [58].

We now prove Theorem 1.2, which shows that conjecture 3.1 holds for fake projective planes admitting an action of the non-abelian group G_{21} of order 21.

According to the Table given in the Appendix there are 6 such surfaces: there are three relevant groups in the table and there are two complex conjugate surfaces for each group [39].

We first prove a general fact about fake projective planes.

Lemma 3.4. *Let S be a fake projective plane with no 3-torsion in $H_1(S,\mathbb{Z})$. Then there exists a unique (ample) line bundle $\mathcal{O}(1)$ such that $K_S \cong \mathcal{O}(3)$.*

Proof. First note that the torsion in $\mathrm{Pic}(S) = H^2(S,\mathbb{Z})$ is isomorphic to $H_1(S,\mathbb{Z})$ (cf Proof of Proposition 2.1), hence $\mathrm{Pic}(S)$ has no 3-torsion by assumption.

By Poincare duality $\mathrm{Pic}(S)/tors \cong H^2(S,\mathbb{Z})/tors$ is a unimodular lattice, therefore there exists an ample line bundle L with $c_1(L)^2 = 1$. Now $K_S - 3c_1(L) \in \mathrm{Pic}(S)$ is torsion which can be uniquely divided by 3. □

Proof of Theorem 1.2. As follows from the classification of the fake projective planes by Prasad-Yeung and Cartwright–Steger, the order of the first homology group of the six fake projective planes with automorphism group G_{21} is coprime to 3 (see the Table in the Appendix). Therefore by Lemma 3.4 we have

$$K_S = \mathcal{O}(3).$$

for a unique line bundle $\mathcal{O}(1)$.

Recall that $G_{21} = Aut(S) = N(\Pi)/\Pi$ where $N(\Pi)$ is a normalizer of Π in $PU(2,1)$ and by [20] the embedding

$$N(\Pi) \subset PU(2,1)$$

lifts to an embedding

$$N(\Pi) \subset SU(2,1)$$

in all cases with G_{21}-action. Therefore $\mathcal{O}_B(-1)$ admits a $N(\Pi)$-linearization and hence $\mathcal{O}(1)$ admits a G_{21}-linearization, compatible with the natural G_{21}-linearization of K_S. We will consider vector spaces $H^*(S,\mathcal{O}(k))$ as G_{21}-representations.

According to Remark 3.3(2), it suffices to show that $H^0(S,\mathcal{O}(2)) = 0$.

We now study the group G_{21} and its representation theory. By Sylow's theorems G_{21} admits a unique subgroup of order 7 and this subgroup is normal. We let σ denote a generator of this subgroup. Let τ denote an element of G_{21} of order 3. Conjugating by τ gives rise to an automorphism of $\mathbb{Z}/7 = \langle\sigma\rangle$ and we can choose τ so that

$$\tau^{-1}\sigma\tau = \sigma^2.$$

Thus G_{21} is a semi-direct product of $\mathbb{Z}/7$ and $\mathbb{Z}/3$ and has a presentation

$$G_{21} = \left\langle \sigma, \tau \mid \sigma^7 = 1, \tau^3 = 1, \sigma\tau = \tau\sigma^2 \right\rangle.$$

Using this presentation it is easy to check that there are five conjugacy classes of elements in G_{21}:

$$\begin{gathered}\{1\}\\ \{\sigma, \sigma^2, \sigma^4\}\\ \{\sigma^3, \sigma^5, \sigma^6\}\\ \{\tau\sigma^k,\ k = 0, \dots, 6\}\\ \{\tau^2\sigma^k,\ k = 0, \dots, 6\}\end{gathered}$$

and by basic representation theory there exist five irreducible representations of G_{21}. Let $d_1, \dots, d_5$ be the dimensions of these representations. Basic representation theory also tells us that each d_i divides 21 and that

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 21.$$

Considering different possibilities one finds the only combination $(d_1, d_2, d_3, d_4, d_5) = (1, 1, 1, 3, 3)$ satisfying the conditions above.

It is not hard to check that the character table of G_{21} is the following one:

	1	$[\sigma]$	$[\sigma^3]$	$[\tau]$	$[\tau^2]$
$\mathbb{C}$	1	1	1	1	1
V_1	1	1	1	ω	$\overline{\omega}$
$\overline{V_1}$	1	1	1	$\overline{\omega}$	ω
V_3	3	b	$\bar{b}$	0	0
$\overline{V_3}$	3	$\bar{b}$	b	0	0

Here we use the notation:

$$\begin{aligned}\omega &= e^{\frac{2\pi i}{3}}\\ \xi &= e^{\frac{2\pi i}{7}}\end{aligned}$$

and

$$b = \xi + \xi^2 + \xi^4 = \frac{-1 + \sqrt{-7}}{2}.$$

Explicitly V_1 and $\overline{V_1}$ are one-dimensional representations restricted from $G_{21}/\langle\sigma\rangle = \mathbb{Z}/3$. V_3 and $\overline{V_3}$ are three-dimensional representations induced from $\mathbb{Z}/7$: $\rho : G_{21} \to GL(V_3)$ is given by matrices

$$\rho(\sigma) = \begin{pmatrix} \xi & & \\ & \xi^2 & \\ & & \xi^4 \end{pmatrix} \quad \rho(\tau) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\overline{V_3}$ is its complex conjugate.

Lemma 3.5. *$H^0(S, \mathcal{O}(4))$ is a 3-dimensional irreducible representation of G_{21} (and thus is isomorphic to V_3 or $\overline{V_3}$).*

Proof. We show that the trace of an element $\sigma \in G_{21}$ of order 7 acting on $H^0(S, \mathcal{O}(4))$ is equal to b or $\bar{b}$. This is sufficient since if $H^0(S, \mathcal{O}(4))$ were reducible it would have to be a sum of three one-dimensional representations and the character table of G_{21} shows that in this case the trace of σ on $H^0(S, \mathcal{O}(4))$ would be equal to 3.

By [38], Proposition 2.4(4) σ has three fixed points P_1, P_2, P_3. Let τ be an element of order 3. τ does not stabilize any of the P_i's, since a tangent space of a fixed point of G_{21} would give a faithful 2-dimensional representation of G_{21} which does not exist as is seen from its character table.

Thus P_i's are cyclically permuted by τ. We reorder P_i's in such a way that

$$\tau(P_i) = P_{i+1 \bmod 3}. \tag{3.1}$$

We apply the so-called Holomorphic Lefschetz Fixed Point Formula (Theorem 2 in [2]) to σ and line bundles $\mathcal{O}(k)$:

$$\sum_{p=0}^{2} (-1)^p \, Tr\big(\sigma\big|_{H^p(S,\mathcal{O}(k))}\big) = \sum_{i=1}^{3} \frac{Tr(\sigma|_{\mathcal{O}(k)_{P_i}})}{(1-\alpha_1(P_i))(1-\alpha_2(P_i))} \tag{3.2}$$

where $\alpha_1(P_i)$, $\alpha_2(P_i)$ are inverse eigenvalues of σ on T_{P_i}:

$$det(1 - t\sigma_*|_{T_{P_i}}) = (1 - t\alpha_1(P_i))(1 - t\alpha_2(P_i)).$$

$\alpha_j(P_i)$ are 7-th roots of unity. We let $\alpha_j := \alpha_j(P_1)$, $j = 1, 2$. Using (3.1) and commutation relations in G_{21} we find that

$$\alpha_j(P_{i+1}) = \alpha_j(P_i)^2$$

so that

$$\begin{aligned} \alpha_j(P_1) &= \alpha_j \\ \alpha_j(P_2) &= \alpha_j^2 \\ \alpha_j(P_3) &= \alpha_j^4. \end{aligned}$$

To find the values of α_j we apply (3.2) with $k = 0$:

$$1 = \frac{1}{(1-\alpha_1)(1-\alpha_2)} + \frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)} + \frac{1}{(1-\alpha_1^4)(1-\alpha_2^4)}. \tag{3.3}$$

All $\alpha_j(P_i)$ are 7-th roots of unity and it turns out that up to renumbering the only possible values of $\alpha_j(P_i)$ which satisfy (3.3) are

$$\begin{aligned} (\alpha_1(P_1), \alpha_2(P_1)) &= (\xi, \xi^3) \\ (\alpha_1(P_2), \alpha_2(P_2)) &= (\xi^2, \xi^6) \\ (\alpha_1(P_3), \alpha_2(P_3)) &= (\xi^4, \xi^5) \end{aligned}$$

or their complex conjugate in which case we would get b instead of $\bar{b}$ for the trace below.

It follows that $Tr(\sigma|_{K_{S,P_i}}) = Tr(\sigma|_{\mathcal{O}(3)_{P_i}})$ is equal to ξ^4, ξ, ξ^2 for $i = 1, 2, 3$ respectively. Dividing by 3 modulo 7 we see that $Tr(\sigma|_{\mathcal{O}(k)_{P_i}})$ is equal to ξ^{6k}, ξ^{5k}, ξ^{3k} for $i = 1, 2, 3$ respectively.

We use (3.2) for $k = 4$ (note that $H^p(S, \mathcal{O}(4)) = 0$ for $p > 0$ by Kodaira vanishing):

$$Tr\big(\sigma\big|_{H^0(S,\mathcal{O}(4))}\big) = \frac{\xi^3}{(1-\xi)(1-\xi^3)} + \frac{\xi^6}{(1-\xi^2)(1-\xi^6)} + \frac{\xi^5}{(1-\xi^4)(1-\xi^5)} = \bar{b}$$

□

We are now ready to show that $H^0(S,\mathcal{O}(2))=0$. Let $\delta = h^0(S,\mathcal{O}(2))$. We know that $h^0(S,\mathcal{O}(4))=3$, hence it follows from Lemma 3.6 applied to $L=L'=\mathcal{O}(2)$ that $\delta \leqslant 2$. Therefore as a representation of G_{21} the space $H^0(S,\mathcal{O}(2))$ is a sum of 1-dimensional representations and the same is true for $H^0(S,\mathcal{O}(2))^{\otimes 2}$. Since $H^0(S,\mathcal{O}(4))$ is three-dimensional irreducible, this implies that the natural morphism

$$H^0(S,\mathcal{O}(2))^{\otimes 2} \to H^0(S,\mathcal{O}(4))$$

has to be zero by Schur's Lemma. Now again by Lemma 3.6 $H^0(S,\mathcal{O}(2))=0$. This finishes the proof of Theorem 1.2. □

Lemma 3.6 (see [43](Lemma 15.6.2))**.** *Let X be a normal and proper variety, L,L' effective line bundles on X. Let*

$$\phi : H^0(X,L)\otimes H^0(X,L') \to H^0(X,L\otimes L')$$

denote the natural map induced by multiplication. Then

$$\dim Im(\phi) \geqslant h^0(X,L)+h^0(X,L')-1.$$

We now consider equivariant derived categories $\mathcal{D}^b_G(S)$ for various subgroups $G\subset G_{21}$. A good reference for equivariant derived categories and their semi-orthogonal decompositions is [22].

It is easy to see that

$$\{\mathcal{O}(-j)\otimes V\}_{j=0,1,2;\; V\in IrrRep(G)} \tag{3.4}$$

forms an exceptional collection in the equivariant derived category $\mathcal{D}^b_G(S)$. We denote by $\mathcal{A}^G_S$ the right orthogonal to this collection.

It is easy to see that the category $\mathcal{A}^G_S$ is non-zero. This follows from the Kuznetsov's criterion (cf the second proof of Corollary 3.2) since the height of the exceptional collection equals $n-1$. We also notice that for any nonzero object A in $\mathcal{A}_S$ the object

$$\bigoplus_{g\in G} g^*A$$

will have a natural G-linearization so will be a non-zero object in $\mathcal{A}^G_S$.

Proposition 3.7. *Let S be a fake projective plane with automorphism group G_{21}. For any $G\subset G_{21}$, $\mathcal{A}^G_S$ is an H-phantom.*

Proof. We denote by Z_G the minimal resolution of S/G. The geometry of Z_G has been carefully studied by Keum [38]: if $|G|=7$ or $|G|=21$ then Z_G is an elliptic surface of Kodaira dimension $\varkappa(Z_G)=1$ (Dolgachev surface), if $|G|=3$ then Z_G is a surface of general type $\varkappa(Z_G)=2$. In each case we compare the equivariant derived category $\mathcal{D}^b_G(S)$ to $\mathcal{D}^b(Z_G)$.

The stabilizers of the fixed points of G action are cyclic and we use [30] or [36] to obtain the semi-orthogonal decomposition

$$\mathcal{D}^b_G(S) \simeq \left\langle \mathcal{D}^b(Z_G), E_1, \ldots, E_{r_G}\right\rangle$$

where r_G is the number of non-special characters of the stabilizers [30].

Note that $p_g(Z_G)=q(Z_G)=0$, therefore

$$\dim HH_*(\mathcal{D}^b(Z_G)) = \dim H^*(Z_G,\mathbb{C}) = \chi(Z_G).$$

We list $\chi(Z_G)$ as well as other relevant invariants in the table:

G	$\#IrrRep(G)$	$Sing(S/G)$	r_G	$\chi(Z_G)$	$\varkappa(Z_G)$
1	1	$\emptyset$	0	3	2
$\mathbb{Z}/3$	3	$3 \times \frac{1}{3}(1,2)$	0	9	2
$\mathbb{Z}/7$	7	$3 \times \frac{1}{7}(1,3)$	9	12	1
G_{21}	5	$3 \times \frac{1}{3}(1,2) + \frac{1}{7}(1,3)$	3	12	1

As already mentioned above r_G is the sum of non-special characters of the stabilizers at fixed points: $\frac{1}{3}(1,2)$ fixed points don't contribute to r_G whereas each $\frac{1}{7}(1,3)$ fixed point has 3 non-special characters.

It follows from the table that in each case we have

$$3 \cdot \#IrrRep(G) = \chi(Z_G) + r_G$$

This implies that the number of exceptional objects in (3.4) matches $\dim HH_*(\mathcal{D}^b_G)$, and therefore in each case $\mathcal{A}^G_S$ is an H-phantom. $\square$

Remark 3.8. When $G = \mathbb{Z}/3$, $r_G = 0$ means that

$$\mathcal{D}^b_G(S) \simeq \mathcal{D}^b(Z_G)$$

in agreement with the derived McKay correspondence [33, 17] which is applicable since S/G has A_2 singularities. Z_G is a fake cubic surface ($p_g(Z_G) = q(Z_G) = 0$, $b_2(Z_G) = 7$) and the image of the exceptional collection (3.4) of 9 objects in $\mathcal{D}^b(Z_G)$ has an H-phantom orthogonal.

Remark 3.9. One can give an alternative proof of Proposition 3.7 using orbifold cohomology. Baranovsky [4] proved an analogue of Hochschild–Kostant–Rosenberg isomorphism for orbifolds. His result implies that (total) Hochschild homology $HH_*(\mathcal{D}^b_G(S))$ is isomorphic as a non-graded vector space to the (total) orbifold cohomology

$$H^*_{orb}(S/G, \mathbb{C}) = \Big(\bigoplus_{g\in G} H^*(S^g, \mathbb{C})\Big)_G = \bigoplus_{[g]\in G/G} H^*(S^g, \mathbb{C})^{Z(g)}.$$

Here S^g is the fixed locus of $g \in G$, $Z(g)$ is the centralizer, $[g]$ is the conjugacy class of g, and $(\cdot)_G$ denotes coinvariants. In our case the following two assumptions are satisfied:

- group G acts trivially on $H^*(S, \mathbb{C})$ (thanks to minimality of S),
- for each element $g \neq 1$ its fixed locus S^g is a union of $\dim H^*(S, \mathbb{C})$ points (this is usually derived from Hirzebruch proportionality principle, see e.g. [39, 38]).

For the so-called main sector $[g] = \{id\}$ we have

$$H^*(S, \mathbb{C})^G = H^*(S, \mathbb{C}) = \mathbb{C}^3.$$

For each $g \neq id$ the fixed locus S^g consists of three points, so $H^*(S^g) = \mathbb{C}^3$ and the action of $Z(g) = \langle g \rangle$ on it is trivial, thus each twisted sector is also 3-dimensional.

Taking the sum over all conjugacy classes $[g]$ we obtain

$$\dim HH_*(\mathcal{D}^b_G(S)) = \dim H^*_{orb}(S/G, \mathbb{C}) = 3 \times \#IrrRep(G),$$

which shows that $HH_*(\mathcal{A}^G_S) = 0$.

Proposition 3.10. *Let S be a fake projective plane with automorphism group G_{21}. In the notations of* [57, 18] *and the appendix assume that the class of S is either $(\mathbb{Q}(\sqrt{-7}), p = 2, \mathcal{T}_1 = \{7\})$ or $\mathcal{C}_{20}$. Let $G = \mathbb{Z}/7 \subset G_{21}$ or $G = G_{21}$. Then the orthogonal to the collection (3.4) in $\mathcal{D}^b_G(S)$ is a K-phantom.*

Proof. Let $\Pi_G \subset \overline{\Gamma}$ be the group generated by $\pi_1(S)$ and G. By [5](0.4) the fundamental group $\pi_1(S/G)$ equals to Π_G/E where $E \subset \Pi_G$ is the subgroup generated by elliptic elements of Π_G i.e. elements $\gamma \in \Pi_G$ such that fixed locus $B^\gamma \neq \emptyset$ is non-empty. Cartwright and Steger in [20] explicitly computed E and so $\pi_1(S/G)$ for various subgroups $\Pi \subset \Pi_G \subset \overline{\Gamma}$ and it turns out that in the cases under consideration the quotients S/G are simply connected. By a standard argument (e.g. using Van Kampen's theorem as in [5](0.5) or [60](Section 4.1), or more generally see [42](Theorem 7.8.1)) the resolutions Z_G are also simply-connected, in particular $H_1(Z_G, \mathbb{Z}) = 0$.

Then $\mathrm{Pic}(Z_G) = H^2(Z_G, \mathbb{Z})$ is finitely generated free abelian group. Keum shows in [38] that Kodaira dimension $\varkappa(Z_G) = 1$ (see also Ishida [29]). Thus Bloch conjecture for Z_G is true by Bloch–Kas–Lieberman [8], that is $CH_0(Z_G) = \mathbb{Z}$. Now by Lemma 2.7 of [26] it follows that $K_0(Z_G)$ is a finitely generated free abelian group, and the same holds for $K_0^G(S) = K_0(\mathcal{D}^b_G(S))$.

The computation of Euler numbers shows that

$$(\text{number of objects in (3.4)}) = \dim HH_*(\mathcal{D}^b_G(S)) = \mathrm{rk}\, K_0(\mathcal{D}^b_G(S)).$$

Finally, the additivity of the Grothendieck group implies that $\mathcal{A}^G_S$ is a K-phantom. □

APPENDIX: AUTOMORPHISMS AND FIRST HOMOLOGY GROUPS OF FAKE PROJECTIVE PLANES

Recall that all fake projective planes S are quotients of a complex ball $B \subset \mathbb{CP}^2$ by a cocompact torsion-free arithmetic subgroup $\Pi = \pi_1(S)$ [57], [18], and each of the fifty possible groups Π corresponds to a pair complex conjugate surfaces S and $\overline{S}$ which are not isomorphic to each other [39]. The first homology group $H_1(S, \mathbb{Z})$ of S is isomorpic to the abelianisation of the $\Pi/[\Pi, \Pi]$ and the automorphism group equals $\mathrm{Aut}(S) = N(\Pi)/\Pi$, where $N(\Pi)$ is the normaliser of Π (in maximal arithmetic group $\overline{\Gamma}$ and hence in any group that contains it, in particular in $PU(2, 1)$).

We enhance the classification table of the fake projective planes given in [18] which is based on GAP and Magma computer code and its output [19] with the automorphism group $Aut(S)$ and the first homology group $H_1(S, \mathbb{Z})$, which we also take from [19].

In the table $\overline{\Gamma}$ is described using the following data: l is a totally complex quadractic extension of a totally real field, p is a prime 2, 3 or 5, $\mathcal{T}_1$ is a set of prime numbers (possibly empty).

N is the index $[\overline{\Gamma} : \Pi]$ and $suf.$ is the suffix (a, b, c, d, e or f) of each group in [19]. G_{21} is the non-abelian group of order 21. In the last column symbol $[n_1, \ldots, n_k]$ denotes the abelian group $(\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_k\mathbb{Z})$.

Consider the quotient-map $f : N(\Pi) \to N(\Pi)/\Pi = \mathrm{Aut}(S)$ and for a subgroup $G \subset \mathrm{Aut}(S) = N(\Pi)/\Pi$ let $\Pi_G \subset N(\Pi) \subset \overline{\Gamma}$ be the preimage $\Pi_G = f^{-1}G$. Line bundle $\mathcal{O}_S(1)$ is G-linearisable $\iff$ group Π_G lifts from $PU(2, 1)$ to $SU(2, 1)$. Computation of Cartwright and Steger [20] shows that it holds for all S and G unless group $\overline{\Gamma}$ lies in classes $\mathcal{C}_2$ or $\mathcal{C}_{18}$. Fundamental group of the quotient-surface $\pi_1(S/G)$ equals Π_G/E where $E \subset \Pi_G$ is the subgroup generated by elliptic elements (cf the proof of Proposition 3.10). All those groups for all S and $G \subset \mathrm{Aut}(S)$ were also computed in [20]: surface S/G is simply-connected in twelve cases, including the four cases of Proposition 3.10.

l or $\mathcal{C}$	p	$\mathcal{T}_1$	N	$\#\Pi$	$suf.$	$\mathrm{Aut}(S)$	$H_1(S,\mathbb{Z})$
$\mathbb{Q}(\sqrt{-1})$	5	$\emptyset$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,4,31]$
		$\emptyset/\{2I\}$			b/b	$\{1\}$	$[2,3,4,4]$
		$\{2\}$	3	1	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,31]$
$\mathbb{Q}(\sqrt{-2})$	3	$\emptyset$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,13]$
		$\emptyset/\{2I\}$			b/b	$\{1\}$	$[2,2,2,2,3]$
		$\{2\}$	3	1	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,13]$
$\mathbb{Q}(\sqrt{-7})$	2	$\emptyset$	21	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
					b	G_{21}	$[2,2,2,2]$
					c	$\{1\}$	$[2,2,3,7]$
		$\{3\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,4,7]$
					b	$\{1\}$	$[2,2,3,4]$
		$\{3,7\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,7]$
					b	$\{1\}$	$[2,3,4]$
		$\{7\}$	21	4	a	G_{21}	$[2,2,2]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,7]$
					d	$\{1\}$	$[2,2,2,3]$
		$\{5\}$	1	1	$-$	$\{1\}$	$[2,2,9]$
		$\{5,7\}$	1	1	$-$	$\{1\}$	$[2,9]$
$\mathbb{Q}(\sqrt{-15})$	2	$\emptyset$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,7]$
					b	$\{1\}$	$[2,2,2,9]$
		$\{3\}$	3	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,3,7]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,2,3]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,3]$
		$\{3,5\}$	3	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[3,7]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,3]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[3]$
		$\{5\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
					b	$\{1\}$	$[2,2,9]$
$\mathbb{Q}(\sqrt{-23})$	2	$\emptyset$	1	1	$-$	$\{1\}$	$[2,3,7]$
		$\{23\}$	1	1	$-$	$\{1\}$	$[3,7]$
$\mathcal{C}_2$	2	$\emptyset$	9	6	a	$(\mathbb{Z}/3\mathbb{Z})^2$	$[2,7]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,7,9]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,9]$
					d	$\mathbb{Z}/3\mathbb{Z}$	$[2,9]$
					f	1	$[2,3,3]$
					g	1	$[2,3,3]$
		$\{3\}$	9	1	$-$	$(\mathbb{Z}/3\mathbb{Z})^2$	$[7]$
$\mathcal{C}_{10}$	2	$\emptyset$	3	1	$-$	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
		$\{17-\}$	3	1	$-$	$\mathbb{Z}/3\mathbb{Z}$	$[7]$
$\mathcal{C}_{18}$	3	$\emptyset$	9	1	a	$(\mathbb{Z}/3\mathbb{Z})^2$	$[2,2,13]$
		$\emptyset/\{2\mathrm{I}\}$	1	1	b/d	1	$[2,3,3]$
		$\{2\}$	3	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,3,13]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,3]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,3]$
$\mathcal{C}_{20}$	2	$\emptyset$	21	1	$-$	G_{21}	$[2,2,2,2,2,2]$
		$\{3-\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,7]$
					b	$\{1\}$	$[2,3,4]$
		$\{3+\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,7]$
					b	$\{1\}$	$[2,3,4]$

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