IPMU 10-0150

G-FANO THREEFOLDS ARE MIRROR-MODULAR

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ABSTRACT. We show that G-Fano threefolds are *mirror-modular* i.e. their *quantum periods* are expansions of weight 2 modular forms in terms of the inverse of Hauptmodul for some genus–zero (moonshine) subgroups of $SL_2(\mathbb{R})$.

Together with Mason's construction this gives a strange correspondence between deformation classes G-Fano threefolds and conjugacy classes of Mathieu group M_{24}.

1. INTRODUCTION

The simplest example of Lian-Yau's *mirror moonshine* for $K3$ surfaces ([16, 17], see also [6, 7, 21, 22]) is the remarkable identity (*modular relation*) of Kachru-Vafa ([14]):

$$\sum_{n\geqslant 0} \frac{(6n)!}{(3n)!n!^3} j(q)^{-n} = E_4(q)^{\frac{1}{2}} \tag{1.1}$$

expressing modular form $E_4(q)$ of weight 4 as square of hypergeometric series $I_1(t) = \sum \frac{6n!}{n!^3 3n!} t^n$ expanded in terms of inverse modular parameter $t = j(q)^{-1}$. Here $\eta(q) = q^{\frac{1}{24}} \prod_{n\geqslant 1}(1-q^n)$ is Dedekind eta-function, $\sigma_3(n) = \sum_{d|n} d^3$, Eisenstein series $E_4(q) = 1 + 240 \sum_{n\geqslant 1} \sigma_3(n) q^n$ equals theta-series θ_{E_8} for lattice E_8, and $j(q) = \frac{E_4^3}{\eta^{24}(q)} = \frac{1}{q} + 744 + 196884q + \dots$ is modular j-invariant.

Smooth anticanonical divisor $S \in |-K_Y|$ in Fano threefold Y is a $K3$ surface endowed with natural lattice polarization $c_1(Y) \in \mathrm{Pic}(Y) \subset \mathrm{Pic}(S) \subset H^2(S,\mathbb{Z}) = II_{3,19}$. Beauville shows inverse ([2]): that generic $K3$ surface lattice-polarized by $c_1(Y) \in Pic(Y)$ appears this way. So Fano threefolds single out 105 (see [13, 19]) out of countably many families of lattice polarized $K3$ surfaces (and also 105 mirror dual families [6]). In fact, almost all moonshine examples listed in [16, 17, 6, 21, 22] appear in that way, and mirror moonshine makes more sense in the context of Fano threefolds. In [12] Golyshev reproduced Iskovskikh's classification of prime Fano threefolds (see e.g. [13]) by effectively combining three elements: mirror, moonshine and minimality.

G-series G_Y (see 2.4) is a certain invariant of Fano variety Y "counting" rational curves on it.

Mirror conjecture (for variations of Hodge structures) states that Laplace transform of G-series for Fano threefold Y is the solution of Picard–Fuchs differential equation for some 1-parameter family of $K3$ surfaces that is called *mirror dual to Y Landau-Ginzburg model.*

Moonshine (*genus-zero modularity*) is explicitly stated as *miraculous eta-product formula*:

$$I_{N,s}(H_{N,c}^{-1}) = \eta(q)^2 \eta(q^N)^2 H_{N,c}^{\frac{N+1}{12}} \tag{1.2}$$

where G_{Y_N} is G-series of Fano threefold Y_N with invariants $H^2(Y_N,\mathbb{Z}) = \mathbb{Z}c_1(Y_N)$, $c_1(Y_N)^3 = 2N$, $I_{N,s} = \mathbb{R}_s G_{Y_N}(t)$ is Laplace transform of G_{Y_N} multiplied by e^{st} for a particular choice of constant $s = s_N$ (see 2.1,2.7), and H_N is a Hauptmodul on Fricke modular curve $X_0(N)/w_N$ (see [5]) with a particular constant term $c = c_N$: $H_{N,c} = \frac{1}{q} + c_N + O(q)$. In case Y is a sextic double solid (i.e. a smooth sextic hypersurface in weighted projective 4-space $\mathbb{P}(1,1,1,1,3)$) we have $N = 1$, $H_1 = j(q)$, $s_1 = 120$, $c_1 = 744$ and formula 1.2 specializes to 1.1 (we present other exact equalities in appendix 7).

Minimality is formalized in the notion of $D3$ *differential equation* (see 5.1). It is a 6-parameter class of differential equations of degree 3, generalizing the construction of regularized quantum differential equations of a Fano threefold from 6 two-point Gromov–Witten invariants.

Modularity conjecture (which is now a theorem) states that function $G = \sum_{n\geqslant 0} a_i t^n$ is G-series of minimal Fano threefold of index one if and only if for some s function $\mathbb{R}_s G$ is of moonshine type (satisfy 1.2 for some N) and is annihilated by differential equation of type $D3$.

This work was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan, Grant-in-Aid for Scientific Research (10554503) from Japan Society for Promotion of Science, Grant of Leading Scientific Schools (N.Sh. 1987.2008.1 and 4713.2010.1) and INTAS 05-100000-8118. This is an update for the second part of my earlier SFB45 preprint [9].

Apart from 17 quantum differential equations of minimal smooth Fano threefolds Golyshev found two more differential equations of type $D3$ and modular origin (5.6 and 5.7) (also these equations were found by Almkvist, van Enckevort, van Straten and Zudilin in [1], and they were already listed in [17]).

It turned out ([9]) that these two examples are quantum differential equations for two deformation classes Y_{28} and Y_{30} of Fano threefolds with $H^2(Y_{28},\mathbb{Z})=\mathbb{Z}^2$, $H^2(Y_{30},\mathbb{Z})=\mathbb{Z}^3$.

Since Fano threefolds Y_{28} and Y_{30} are not minimal one naively expects their regularized quantum differential equations to be of degree 4 and 5, but these varieties occur to be *quantum minimal* — minimal differential equation vanishing $\hat{G}$-series of these varieties has degree 3 (see [10] for the details).

In [9] we made an observation that both Y_{28} and Y_{30} are *G-Fano threefolds* i.e. for some complex structure they admit a finite group action $G:Y$ with $Pic^G(Y)=\mathbb{Z}c_1(Y)$. In [10] we shown that G-Fano threefolds are quantum minimal. So it is natural to look whether other G-Fano threefolds are mirror-modular.

Families Y_{28} and Y_{30} are two of total eight families of G-Fano threefolds (see [20]). In these article we will show that all 8 families of G-Fano threefolds are mirror-modular.

2. Preliminaries

2.1. **Shifts and regularizations.** For a number s and a power series $A=\sum_{n\geqslant 0}a_it^n$ define its regularization (Laplace transform $\mathcal{L}$), inverse Laplace transform $\mathcal{L}^{-1}$, shifted regularization $\mathcal{L}_s$, regular shift $\mathcal{S}_g$ and normalization $\mathcal{N}$ by the formulas

$$\hat{A}=\mathcal{L}A=\sum(a_i\cdot n!)t^n,$$
$$\mathcal{L}^{-1}A=\sum\frac{a_i}{n!}t^n,$$
$$\mathcal{L}_sA=\mathcal{L}(e^{s\cdot t}\cdot A)$$
$$\mathcal{S}_sA=\mathcal{L}_s\mathcal{L}^{-1}A$$
$$\mathcal{N}A=S_{-a_1}A$$

2.2. **Fano varieties.** [see e.g. [13]]

Let Y be a Fano variety — smooth variety with ample anticanonical class ω_Y^{-1}.

Y is simply-connected, by Kodaira vanishing $H^i(Y,\mathcal{O}_Y)=0$ for $i>0$, so $c_1:\mathrm{Pic}(Y)\to H^2(Y,\mathbb{Z})$ is an isomorphism and both are isomorphic to $\mathbb{Z}^\rho$, where ρ is called Picard number. Lefschetz pairing on $H^2(Y,\mathbb{Z})$ defined by $(A,B)=\int_{[Y]}A\cup B\cup c_1(Y)^{\dim Y-2}$ is nondegenerate (by Hard Lefschetz theorem), so $H^2(Y,\mathbb{Z})$ is a lattice and we denote its discriminant by $d(Y)$.

Anticanonical *degree* of Fano variety Y is $deg(Y)=(c_1(Y),c_1(Y))=\int_{[Y]}c_1(Y)^{\dim Y}$. Euler number is a topological Euler characteristic $\chi(Y)=\int_{[Y]}c_{\dim Y}$.

Fano index $r(Y)$ is divisibility of $c_1(Y)$ in the lattice $H^2(Y,\mathbb{Z})$ i.e. $r(Y)=\max\{r\in\mathbb{Z}|c_1(X)=rH,H\in H^2(Y,\mathbb{Z})\}$.

Definition 2.1 ([20])**.** Fano variety Y with group action $G:Y$ is called G-Fano if $H^2(Y,\mathbb{Z})^G=\mathbb{Z}$.

2.3. **Quantum differential equations, G-series and Givental's constant.** Let $\star$ be quantum multiplication on cohomologies of Y defined by

$$\int_{[Y]}(\gamma_1\star\gamma_2)\cup\gamma_3=\sum_{d\geqslant 0}\langle\gamma_1,\gamma_2,\gamma_3\rangle_d\,t^d \tag{2.2}$$

where closed genus 0 3-point correlator $\langle\gamma_1,\gamma_2,\gamma_3\rangle_d=\int_{\overline{\mathcal{M}}_{0,3}(Y,d)}\prod ev_i^*(\gamma_i)$ is a Gromov-Witten invariant "counting" maps $f:\mathbb{P}^1\to Y$ of degree $d=\int_{[\mathbb{P}^1]}f^*c_1(Y)$ passing through homology classes Poincare-dual to γ_i.

Take $D=t\frac{d}{dt}$ and define *quantum differential equation* as a trivial vector bundle over Spec $\mathbb{C}[t,t^{-1}]$ with fibre $H^*(Y)$ and connection

$$D\Phi=c_1(Y)\star\Phi \tag{2.3}$$

where $\Phi\in H^*(Y)[[t]]$.

Let $\mathcal{G}_Y(t) = [pt] + \sum_{n\geqslant 1} \mathcal{G}^{(n)} t^n$ be the unique analytic solution of 2.3 starting with class Poincare-dual to the the class of the point, and define G-series as

$$G_Y(t) = \int_{[Y]} \mathcal{G}_Y(t) = 1 + \sum_{n\geqslant 1} G^{(n)} t^n \tag{2.4}$$

We note that the first coefficient $G^{(1)} = \langle [pt], c_1(Y), [Y] \rangle_1 = \int_{|t|=\epsilon} (\int_{[Y]} c_1(Y) \star [pt]) \frac{dt}{t^2}$ is zero according to the String equation, and we name $G^{(2)} = \langle [pt] \rangle_2$ (the expected number of anticanonical conics passing through a point) as *Givental's constant* $G(Y)$, so

$$G_Y = 1 + G(Y) \cdot t^2 + O(t^3). \tag{2.5}$$

Define $\hat{G}$-series (also known as *the quantum period* of Y) as

$$\hat{G}_Y = \hat{G_Y} = 1 + \sum_{n\geqslant 1} n! \cdot G^{(n)} t^n = 1 + 2G(Y) \cdot t^2 + \dots \tag{2.6}$$

Conjecturally $\hat{G}_Y$ should have integer coefficients.

For convenience we define I-series to be a regularized shifted G-series:

Definition 2.7. For a given number s define $I_{Y,s} = \mathcal{L}_s G_Y$, in particular $I_{Y,0} = \hat{G}_Y$. By a slight abuse of notation we will say that power series $I(t) = 1 + \sum_{n\geqslant 1} i^{(n)} t^n$ is a regular I-series of smooth Fano variety Y if $\mathcal{N} I = \hat{G}_Y$ i.e. $I = I_{Y, i^{(1)}}$.

3. G-Fano threefolds and A-model G-series

There are 8 deformation classes of G-Fano threefolds Y with $\operatorname{rk} H^2(Y,\mathbb{Z}) > 1$ (see e.g. [20]). Two of them has Fano index two, these are $Y_{48}^{(3)} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $Y_{48}^{(2)} = W \subset \mathbb{P}^2 \times \mathbb{P}^2$ of degree 48. Six other have Fano index one: Y_{30} of degree 30, Y_{28} of degree 28, Y_{24} of degree 24, Y_{20} of degree 20, $Y_{12}^{(2)}$ and $Y_{12}^{(3)}$ of degree 12.

In this section we are going to describe all of them geometrically. The details regarding the computation of the respective quantum periods can be found in [3].

Definition 3.1. Fano threefold $Y_{48}^{(2)}$ is the variety $Fl(1,2,3)$ of complete flags in $\mathbb{P}^2$ i.e. a hyperplane section of Segre fourfold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. This variety is unique in its deformation class number 32 in table 2 of [19], it has Fano index 2, degree 48, $\chi = 6$ and $\rho = 2$.

This variety can also be described as a projectivization of tangent bundle on $\mathbb{P}^2$ or variety of complete flags in $\mathbb{P}^2$.

Corollary 3.2. *I-series $I_{6,2;2} = \mathbb{R} G_{Fl(1,2,3)}$ is given by the pullback of hypergeometric series from two-dimensional torus*

$$I_{6,2;2} = \sum_{a,b\geqslant 0} \frac{(a+b)!(2a+2b)!}{a!^3 b!^3} t^{2(a+b)} = 1 + 4t^2 + 60t^4 + 1120t^6 + 24220t^8 + 567504t^{10} + \dots \tag{3.3}$$

Proof. Combine the computation of I-series of toric variety $\mathbb{P}^2 \times \mathbb{P}^2$ in [11] and quantum Lefschetz principle in [4]. □

Definition 3.4. Fano threefold $Y_{48}^{(3)}$ is just a Cartesian cube of a line i.e. Segre threefold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$. This variety is unique in its deformation class number 27 in table 3 of [19], it has Fano index 2, degree 48, $\chi = 8$ and $\rho = 3$.

Corollary 3.5. *I-series $I_{6,3;2} = \mathbb{R} G_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$ is given by the pullback of hypergeometric series from three-dimensional torus*

$$I_{6,3;2} = \sum_{a,b,c\geqslant 0} \frac{(2a+2b+2c)!}{a!^2 b!^2 c!^2} t^{2(a+b+c)} = 1 + 6t^2 + 90t^4 + 1860t^6 + 44730t^8 + 1172556t^{10} + \dots \tag{3.6}$$

Remark 3.7. Threefolds $Y_{48}^{(2)}$ and $Y_{48}^{(3)}$ has isomorphic hyperplane sections — del Pezzo surface of degree 6. This implies the relation $G_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(\sqrt{t}) = G_{Fl(1,2,3)}(\sqrt{t}) \cdot e^t$

Definition 3.8. Fano threefold Y_{30} is the blowup of a curve of bidegree $(2,2)$ on $Fl(1,2,3) \subset \mathbb{P}^2 \times \mathbb{P}^2$. This deformation class of varieties has number 13 in table 3 of [19], it has degree 30, $\chi = 8$ and $\rho = 3$.

Proposition 3.9. *Y_{30} is a complete intersection of three numerically effective divisors of tridegrees $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.*

Corollary 3.10. *I-series $I_{15} = \mathcal{L}_3 G_{Y_{30}}$ is given by the pulback of hypergeometric series from three-dimensional torus*

$$I_{15} = \sum_{a,b,c\geqslant 0} \frac{(a+b)!(a+c)!(b+c)!(a+b+c)!}{a!^3 b!^3 c!^3} t^{a+b+c} = 1 + 3t + 15t^2 + 105t^3 + 855t^4 + 7533t^5 + \ldots \tag{3.11}$$

Let Q be 3-dimensional quadric.

Definition 3.12. Let Y_{28} be the blowup of a twisted quartic on Q. This deformation class of varieties has number 21 in table 2 of [19], it has degree 28, $\chi = 6$ and $\rho = 2$.

Denote the I-series of Y_{28} as $I_{14} = \hat{G}_{Y_{28}}$.

Definition 3.13. Fano threefold Y_{24} is a hyperplane section of Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^{15}$. This deformation class of varieties has number 1 in table 4 of [19], it has degree 24, $\chi = 2$ and $\rho = 4$.

Corollary 3.14. *I-series $I_{12} = \mathcal{L}_4 G_{Y_{24}}$ is given by the pullback of hypergeometric series from four-dimensional torus*

$$I_{12} = \sum_{a,b,c,d\geqslant 0} \frac{(a+b+c+d)!^2}{a!^2 b!^2 c!^2 d!^2} t^{a+b+c+d} = 1 + 4t + 28t^2 + 256t^3 + 2716t^4 + 31504t^5 + \ldots \tag{3.15}$$

Definition 3.16. Let Y_{20} be the blowup of projective space $\mathbb{P}^3$ with center a curve of degree 6 and genus 3 which is an intersection of cubics. This deformation class of varieties has number 12 in table 2 of [19], it has degree 20, $\chi = 0$ and $\rho = 2$.

Proposition 3.17. *Y_{20} is an intersection of Segre variety $\mathbb{P}^3 \times \mathbb{P}^3$ by linear subspace of codimension 3.*

Corollary 3.18. *I-series $I_{10} = \mathcal{L}_2 G_{Y_{20}}$ is given by the pulback of hypergeometric series from two-dimensional torus*

$$I_{20} = \sum_{a,b\geqslant 0} \frac{(a+b)!^4}{a!^4 b!^4} t^{a+b} = 1 + 2t + 18t^2 + 164t^3 + 1810t^4 + 21252t^5 + 263844t^6 + 3395016t^7 + \ldots \tag{3.19}$$

Definition 3.20. Fano threefold $Y_{12}^{(2)}$ can be described either as a section of Segre fourfold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ by quadric or as double cover of W with branch locus in anticanonical divisor. This deformation class of varieties has number 6 in table 2 of [19], it has degree 12, $\chi = -12$ and $\rho = 2$.

Corollary 3.21. *I-series $I_{6,2} = \mathcal{L}_4 G_{Y_{12}^{(2)}}$ is given by the pullback of hypergeometric series from two-dimensional torus*

$$I_{6,2} = \sum_{a,b\geqslant 0} \frac{(a+b)!(2a+2b)!}{a!^3 b!^3} t^{a+b} = 1 + 4t + 60t^2 + 1120t^3 + 24220t^4 + 567504t^5 + \ldots \tag{3.22}$$

Remark 3.23. Series $I_{6,2}$ and $I_{6,2;2}$ are related by change of coordinates $I_{6,2;2}(t) = I_{6,2}(t^2)$.

Definition 3.24. Fano threefold $Y_{12}^{(3)}$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with branch locus in anticanonical divisor. This deformation class of varieties has number 1 in table 3 of [19], it has degree 12, $\chi = -8$ and $\rho = 3$.

Corollary 3.25. *I-series $I_{6,3} = \mathcal{L}_6 G_{Y_{12}^{(3)}}$ is given by the pullback of hypergeometric series from two-dimensional torus*

$$I_{6,3} = \sum_{a,b,c\geqslant 0} \frac{(2a+2b+2c)!}{a!^2 b!^2 c!^2} t^{a+b+c} = 1 + 6t + 90t^2 + 1860t^3 + 44730t^4 + 1172556t^5 + \ldots \tag{3.26}$$

Remark 3.27. Series $I_{6,3}$ and $I_{6,3;2}$ are related by change of coordinates $I_{6,3;2}(t) = I_{6,3}(t^2)$.

4. Eta-products, Hauptmoduln and their M-series

Let $\eta(q)$ be Dedekind's eta-function: $\eta(q) = q^{\frac{1}{24}} \prod_{n\geqslant 1}(1-q^n)$.

Consider 4 eta-products and 1 eta-quotient:

$$\eta_{6+} = \eta(q)\eta(q^2)\eta(q^3)\eta(q^6) \tag{4.1}$$

$$\eta_{10+} = \eta(q)\eta(q^2)\eta(q^5)\eta(q^{10}) \tag{4.2}$$

$$\eta_{12+} = \frac{\eta(q^2)^4\eta(q^6)^4}{\eta(q)\eta(q^3)\eta(q^4)\eta(q^{12})} \tag{4.3}$$

$$\eta_{14+} = \eta(q)\eta(q^2)\eta(q^7)\eta(q^{14}) \tag{4.4}$$

$$\eta_{15+} = \eta(q)\eta(q^3)\eta(q^5)\eta(q^{15}) \tag{4.5}$$

Let $\sigma_1(n)$ be -24 times valuation of η_{n+}.

For a constant c and a conjugacy class g of Monster simple group denote by $T_{g,c}$ its McKay-Thompson series (see [5]) with constant term normalized to be c: $T_{g,c} = \frac{1}{q} + c + \sum_{n\geqslant 1} a_i(g)q^n$.

Take $H_{6A,2} = T_{6A,10}$, $H_{6A,3} = T_{6A,14}$, $H_{10A} = T_{10A,4}$, $H_{12A} = T_{12A,6}$, $H_{14A,s} = T_{14A,s}$, $H_{15A,s} = T_{15A,s}$:

$$T_{6A,0} = \frac{1}{q} + 79q + 352q^2 + 1431q^3 + 4160q^4 + 13015q^5 + 31968q^6 + \dots \tag{4.6}$$

$$H_{10A} = 8 + \frac{\eta^4(q)\eta^4(q^5)}{\eta^4(q^2)\eta^4(q^{10})} + 16\frac{\eta^4(q^2)\eta^4(q^{10})}{\eta^4(q)\eta^4(q^5)} = \frac{1}{q} + 4 + 22q + 56q^2 + 177q^3 + 352q^4 + \dots \tag{4.7}$$

$$H_{12A} = \left(\frac{\eta(q^2)^2\eta(q^6)^2}{\eta(q)\eta(q^3)\eta(q^4)\eta(q^12)}\right)^6 = \frac{1}{q} + 6 + 15q + 32q^2 + 87q^3 + 192q^4 + \dots \tag{4.8}$$

$$H_{14A} = 4 + \frac{\eta^3(q)\eta^3(q^7)}{\eta^3(q^2)\eta^3(q^{14})} + 8\frac{\eta^3(q^2)\eta^3(q^{14})}{\eta^3(q)\eta^3(q^7)} = \frac{1}{q} + 1 + 11q + 20q^2 + 57q^3 + 92q^4 + \dots \tag{4.9}$$

$$H_{15A} = 3 + \frac{\eta^2(q)\eta^2(q^5)}{\eta^2(q^3)\eta^2(q^{15})} + 9\frac{\eta^2(q^3)\eta^2(q^{15})}{\eta^2(q)\eta^2(q^5)} = \frac{1}{q} + 1 + 8q + 22q^2 + 42q^3 + 70q^4 + \dots \tag{4.10}$$

Define $M_n(t)$ as power-series satisfying the functional equation $M_n(\frac{1}{H_n(q)}) = \eta_{n+} \cdot H_n^{\frac{\sigma_1(n)}{24}}$:

$$M_{6,2}(H_{6A,2}^{-1}(q)) = \eta_{6+} \cdot H_{6A,2}^{\frac{1}{2}} \tag{4.11}$$

$$M_{6,3}(H_{6A,3}^{-1}(q)) = \eta_{6+} \cdot H_{6A,3}^{\frac{1}{2}} \tag{4.12}$$

$$M_{10}(H_{10}^{-1}(q)) = \eta_{10+} \cdot H_{10A}^{\frac{3}{4}} \tag{4.13}$$

$$M_{12}(H_{12}^{-1}(q)) = \eta_{12+} \cdot H_{12A}^{\frac{1}{2}} \tag{4.14}$$

$$M_{14,s}(H_{14,s}^{-1}(q)) = \eta_{14+} \cdot H_{14A,s} \tag{4.15}$$

$$M_{15,s}(H_{15,s}^{-1}(q)) = \eta_{15+} \cdot H_{15A,s} \tag{4.16}$$

5. Differential equations and their solutions

Let t be a coordinate on $G_m = \text{Spec }\mathbb{C}[t,t^{-1}]$ and $D = t\frac{d}{dt}$.

Definition 5.1 ([12]). *Normalized operators of type* $D3$ is the following 5-dimensional family of differential operators depending on parameters b_1, b_2, b_3, b_4, b_5 [1]:

$$\begin{aligned} L(b_1,b_2,b_3,b_4,b_5) = D^3 - t\cdot b_1 D(D+1)(2D+1) - t^2\cdot(D+1)(b_2 D(D+2) + 4b_3) - \\ - t^3\cdot b_4(D+1)(D+2)(2D+3) - t^4\cdot b_5(D+1)(D+2)(D+3) \end{aligned}$$

[1] Original definition has the other basis $a_{01}, a_{02}, a_{03}, a_{11}, a_{12}$ for parameter space $\mathbb{A}^5$. Bases a and b are equivalent over $\mathbb{Z}$: $b_1 = a_{11}$, $b_2 = a_{12} + 2a_{01} - a_{11}^2$, $b_3 = a_{01}$, $b_4 = a_{02} - a_{01}a_{11}$, $b_5 = a_{03} - a_{01}^2$; $a_{01} = b_3$, $a_{02} = b_4 + b_1b_3$, $a_{03} = b_5 + b_3^2$, $a_{11} = b_1$, $a_{12} = b_2 - 2b_3 + b_1^2$.

Define $L_{6,2}$, $L_{6,3}$, L_{10}, L_{12}, L_{14}, L_{15} as follows:

$$L_{6,2} = L(6, 368, 88, 1056, 3584) \tag{5.2}$$
$$L_{6,3} = L(8, 360, 108, 864, 2160) \tag{5.3}$$
$$L_{10} = L(2, 112, 28, 184, 336) \tag{5.4}$$
$$L_{12} = L(2, 80, 24, 96, 0) \tag{5.5}$$
$$L_{14} = L(1, 59, 16, 68, 80) \tag{5.6}$$
$$L_{15} = L(1, 43, 12, 78, 216) \tag{5.7}$$

Locally equation L_n has 1-dimensional space of analytic solutions spanned by F_n:

$$F_{6,2}(t) = 1 + 44t^2 + 528t^3 + 11292t^4 + 228000t^5 + 4999040t^6 + 112654080t^7 + \dots \tag{5.8}$$
$$F_{6,3}(t) = 1 + 54t^2 + 672t^3 + 15642t^4 + 336960t^5 + 7919460t^6 + 191177280t^7 + \dots \tag{5.9}$$
$$F_{10}(t) = 1 + 14t^2 + 72t^3 + 882t^4 + 8400t^5 + 95180t^6 + 1060080t^7 + \dots \tag{5.10}$$
$$F_{12}(t) = 1 + 12t^2 + 48t^3 + 540t^4 + 4320t^5 + 42240t^6 + 403200t^7 + \dots \tag{5.11}$$
$$F_{14}(t) = 1 + 8t^2 + 24t^3 + 240t^4 + 1440t^5 + 11960t^6 + 89040t^7 + \dots \tag{5.12}$$
$$F_{15}(t) = 1 + 6t^2 + 24t^3 + 162t^4 + 1080t^5 + 7620t^6 + 55440t^7 + \dots \tag{5.13}$$

6. Equivalence of realizations

Lemma 6.1. *I-series I_n are solutions to differential equations L_n listed in 5, i.e. $\mathcal{N} I_n = F_n$.*

Proof. By [10] G-function G_V of G-Fano threefold V satisfy ODE of order 4 and its Fourier-Laplace transform $\hat{G}_V$ satisfy a Fuchsian ODE of order 3. □

Lemma 6.2. *M-series M_n are solutions to differential equations L_n listed in 5, i.e. $\mathcal{N} M_n = F_n$.*

Proof. By Proposition 21 of [23] function M_n satisfy some differential equation of order 3 in $D = t\frac{d}{dt}$. Moreover □

We have an immediate

Corollary 6.3. *For every n series F_n, $\mathcal{N} I_n$ and $\mathcal{N} M_n$ coincide.*

and it implies the main

Theorem 6.4. *For every n there are constants s_n and c_n such that I-series of G-Fano threefolds satisfy generalized miraculous eta-product formula 1.2: $I_{n,s}(H_{n,c}^{-1}) = \eta_{n+} \cdot H_{n,c}^{\frac{\sigma_1(n)}{24}}$.*

7. Golyshev's modularity of minimal Fano threefolds

Let Y_N be a Fano threefold with $H^2(Y, \mathbb{Z}) = \mathbb{Z}K_Y$ and $(-K_Y)^3 = 2N$. Let $G_Y = 1 + \sum_{n \geqslant 2} G^{(n)}(Y)t^n$ be its G-series. Golyshev's modularity conjecture states that for every $N = 1, \dots, 9, 11$ there exists such a constant s_N, and a Monster conjugacy class g_N ($N + N$ in notations of [5]), and a constant c_N such that

$$\eta^2(q)\eta^2(q^N) \cdot T_{g_N,c_N}^{\frac{N+1}{12}}(q) = I_{Y_N,s_N}\left(\frac{1}{T_{g_N,c_N}(q)}\right)$$

where $T_{g_N,c_N} = \frac{1}{q} + c_N + O(q)$ is McKay-Thompson series for conjugacy class g_N with constant term c_N.

For $N = 6$ the conjugacy class $6 + 6$ is $6B$, for other values of N it is NA.

In the table we specify values N, $g = g_N$, $c = c_N$ and $s = s_N$ for 16 G-Fano threefolds of index 1 — 10 Golyshev's cases and 6 other G-Fano threefolds with $N = 6, 6, 10, 12, 14, 15$ that are explained in this paper.

For integer N we define Euler number $\phi(N) = N\prod_{p|N}(1 - p^{-1})$, $\psi(N) = N\prod_{p|N}(1 + p^{-1})$, $\epsilon(N) = \frac{24}{\psi(N)}$ and $\iota(N) = \sum_{M|N} \phi(N)\psi(N)$.

N	1	2	3	4	5	6	6	6	7	8	9	10	11	12	14	15
$\epsilon(N)$	24	8	6	4	4	2	2	2	3	2	2	0∗	2	1	1	1
$\iota(N)$	24	16	12	10	8	8	8	8	6	6	4	8∗	4	5∗	4	4
s	120	24	12	8	6	6	5	4	4	4	3	2	*	4	*	*
c	744	104	42	24	16	14	12	10	9	8	6	4	s+2	6	s+1	s+1
g	1A	2A	3A	4A	5A	6A	6B	6A	7A	8A	9A	10A	11A	12A	14A	15A
ρ	1	1	1	1	1	3	1	2	1	1	1	2	1	4	2	3

Proposition 7.1. *Number $\psi(N)$ equals to index of $\Gamma_0(N)$ in $SL(2,\mathbb{Z})$. Number $\phi(N)$ equals to index of $\Gamma_1(N)$ in $\Gamma_0(N)$.*

Number $\epsilon(N)$ is integer only for 15 values of N.

Proposition 7.2. *Let N be one of $1,\dots,8,11,14,15,23$. Denote by M_{23} the Mathieu group and by V its natural 24-dimensional representation induced from Mathieu group M_{24} (which in turn induced from Conway group Co_1). Let $g \in M_{23}$ be an element of Mathieu group M_{23} of order N. Then number $\epsilon(N)$ equals to the trace $Tr_V g$ and number $\iota(N)$ equals to dimension of invariants $\dim V^g$.*

Remark 7.3. Note that for $N = 11, 14, 15$ the respective space of modular forms is 2-dimensional and any choice of s produces a modular relation. There is a particular choice of c depending on s: the difference $(c - s)$ is an invariant of Fano threefold. For $N \neq 11, 14, 15$ the choice of s is unique: s is a natural number such that I-function has singularity at 0[2].

8. Mathieu groups.

Definition 8.1. Let S be the set of 24 points and $S_{24} = Aut(S)$ is its group of automorphisms. Let $M = S^{\mathbb{Q}}$ be a vector space of the tautological 24-dimensional representation of S_{24}. Mathieu group M_{24} is a particular simple subgroup of S_{24} of order $244823040 = 23 \cdot 11 \cdot 7 \cdot 5 \cdot 3^3 \cdot 2^{10} = 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot (16 \cdot 3)$.

Natural action $M_{24} : S$ is 5-transitive.

Definition 8.2. Stabilizer of one point in this action is simple Mathieu group M_{23} of order $23 \cdot 11 \cdot 7 \cdot 5 \cdot 3^2 \cdot 2^7$.

Definition 8.3. Any transposition $h \in S_{24}$ of symmetric group decomposes into the product of cycles, so we will say *frame shape* of h is $\prod_{n \geqslant 1} \mathbf{i}^{a_i}$ where $a_i(h)$ is number of cycles of length n. Frame shape is a complete invariant for conjugacy classes of S_{24} (and complete up to power-equivalence for M_{24} and M_{23}).

Example 8.4.
1. S_{24} has 1575 frame shapes and conjugacy classes (equal to number of partitions of 24),
2. M_{24} has 21 frame shapes and 26 conjugacy classes,
3. M_{23} has 12 frame shapes and 17 conjugacy classes.

Definition 8.5. Order of element (conjugacy class, frame shape) h is equal to the least common multiple of all cycle's lengths: $n(h) = lcm(a_1, \dots, a_{24})$. Denote by $G(h) = Trh_{|\mathbb{Q}^{24}}$ the number a_1 of cycles of length one (Lefschetz fixed point formula on finite set of 24 elements).

Obviously $G(h)$ is integer non-negative and if $h \in M_{23}$ then $G(h) \geqslant 1$.

Proposition 8.6 (Frobenius, Mukai)**.**
1. *$h \in M_{24}$ comes from M_{23} $\iff$ $G(h) \geqslant 1$.*
2. *For element $h \in M_{23}$ number $G(h)$ depends only on $n(h)$: $G(h) = \epsilon(n)$.*
3. *There are 12 orders of elements in M_{23}: from 1 to 8, 11, 14, 15 and 23. Frame shapes of M_{23} are determined by the orders.*

Proposition 8.7. *M_{23} has the following 12 frame shapes:*

g	1^{24}	$1^8 2^8$	$1^6 3^6$	$1^4 2^2 4^4$	$1^4 5^4$	$1^2 2^2 3^2 6^2$	$1^3 7^3$	$1^2 2^1 4^1 8^2$	1^2
n	1	2	3	4	5	6	7	8	1
w	12	8	6	5	4	4	3	3	[illegible]

M_{24} has the following 9 extra frame shapes:

g	2^{12}	3^8	$2^4 4^4$	4^6	6^4	$2^2 10^2$	$2^1 4^1 6^1 12^1$	12^2	$3^1 21^1$
n	2	3	4	4	6	10	12	12	21
N	4	9	8	16	36	20	24	144	63
w	6	4	4	3	2	2	2	1	1

[2]There is a single ambiguity in case $N = 7$: one have to choose $s = 4$ but not $s = 5$.

9. Mason's cusp-forms.

Given a frame shape $g = \prod \mathbf{i}^{a_i}$ consider a function $\eta_g = \prod \eta(q^n)^{a_i}$, where $\eta(q) = q^{\frac{1}{24}} \prod_{m\geqslant 1}(1-q^m)$ is Dedekind's eta-function.

Define *weight* of frame shape as $w(g) = \frac{\sum a_i}{2}$ and *level* as $N(g) = gcd(a_1, \ldots, a_{24}) \cdot lcm(a_1, \ldots, a_{24})$. [3]

Theorem 9.1 (Mason [18])**.** *Let g be one of 21 frame shapes of M_{24}. Then η_g is a cusp-form and Hecke-eigenform of weight $w(g)$ and level $N(g)$ with quadratic nebentypus character (if weight $w(g)$ is even then the character is trivial). Moreover, all these functions η_g form a character of a particular graded representation of M_{24} functorially constructed from M.*

Theorem 9.2 (Dummit,Kisilevsky,McKay [8];Koike [15]; Mason [18])**.** *Only for 30 out of 1575 frame shapes g of S_{24} the respective eta-product η_g is a Hecke eigen-cuspform. There are 2 extra frame shapes with non-integer weight (24^1 and 8^3) and 7 extra frame shapes with integer weight:*

g	$3^2 9^2$	$4^2 8^2$	$2^3 6^3$	$2^1 22^1$	$4^1 20^1$	$6^1 18^1$	$8^1 16^1$
n	9	8	6	22	20	18	16
N	27	32	12	44	80	108	128
w	2	2	3	1	1	1	1

It is known all of them come as characters of extension $2^{11}\dot{M}_{24}$.

10. Symplectic automorphisms of $K3$ surfaces.

Theorem 10.1 (Nikulin)**.** *Let g be an automorphism of finite order N on $K3$ surface S preserving the holomorphic volume form ω: $g^N = 1$, $g^*\omega = \omega$. Denote by $F(g)$ the number of its fixed points: $F(g) = Trg_{|H^\bullet(S,\mathbb{Q})}$. (Lefschetz fixed point formula on $K3$). Then*

1. *Order of symplectic automorphism is bounded by $N \leqslant 8$.*
2. *$F(g)$ depends only on the order and $F(g) = \epsilon(N)$.*

Theorem 10.2 (Mukai)**.** *Finite group G acts on $K3$ surface S preserving the holomorphic volume form ω $\iff$ the following two conditions are satisfied:*

1. *G can be embedded in M_{23}*
2. *tautological action of G on set S has at least 5 orbits, or in other words — the dimension of invariants of representation $H^0(S,\mathbb{Q})$ is at least 5*

Problem 10.3. *Is there any direct geometric relation between G-Fano threefolds and symplectic automorphisms of $K3$ surfaces?*

11. Measuring rationality

Definition 11.1. We say that deformation class of smooth Fano threefolds is of *irrational* type if there is at least one irrational variety in this family. Otherwise we say it is of *rational type*.

Proposition 11.2. *All G-Fano threefolds of higher rank are rational.*

Proof. Case by case. Index 2:

1. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is obviously rational
2. Projection from $W = X_{(1,1)} \subset \mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^2$ along the first factor is birational.

Index 1:

1. Projection from $Y_{15} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^2$ (contract the first $\mathbb{P}^2$ and project from a point on the second $\mathbb{P}^2$ factor) is birational. Also Y_{15} is known to be blowup of W.
2. Projection from $Y_{14} \subset Q \times Q$ to one of the factors is birational (inverse to the blowup of a twisted quartic).
3. Projection from $Y_{12} = X_{(1,1,1,1)} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is birational.
4. Projection from $Y_{10} = X_{(1,1),(1,1),(1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^3$ to one of the factors is birational.
5. Varieties $Y_{6,2}$ are divisors of type $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Projection to the first factor is a conic bundle over $\mathbb{P}^2$ with degeneracy locus of degree six.
6. Varieties $Y_{6,3}$ are covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched in anticanonical divisor. Composition of the double cover and projection to the product of first two factors gives a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.

[3]In our examples it will be $\min\{i | a_i \neq 0\} \cdot \max\{i | a_i \neq 0\}$.

□

Proposition 11.3. *Let X_N be a deformation class G-Fano threefolds of index $r = 1$ and degree $2N$. Then X_N is of rational type $\iff \epsilon(N) \geqslant 2$.*

Proof. Case by case. Combine proposition 11.2 and tables in the end of [13]. □

Acknowledgement. Author thanks Tom Coates, Alessio Corti, Vasily Golyshev, Shinobu Hosono, Ludmil Katzarkov, Bumsig Kim, Yuri Prokhorov, Duco van Straten, Todor Milanov, Kyoji Saito, Maria Vlasenko, Noriko Yui for useful discussions.

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