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MINIFOLDS 1: ON THE PROJECTIVE FOURSPACE

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Abstract. We prove that any fourfold with a full exceptional collection of five coherent sheaves is isomorphic to the projective space. This is the first in a series of papers on *minifolds*.

1. Introduction.

We say that a complex projective manifold X is a *topologically minimal manifold* if its cohomology $H^\bullet(X,\mathbb{Q})$ is $(\dim X+1)$-dimensional (such manifolds are also called $\mathbb{Q}$-*homology projective spaces*). We say that X is a *homologically minimal manifold* if there is a full exceptional collection of $(\dim X+1)$ objects in its bounded derived category of coherent sheaves $\mathcal{D}^b_{coh}(X)$. In what follows we nickname them as *minifolds*. The study of minifolds was initiated by Bondal [2] and some general results were obtained by Bondal–Polishchuk [5] and Positselski [22].

The point is the atomic minifold — the derived category $\mathcal{D}^b_{coh}(pt)$ is just the derived category of vector spaces, it is generated by one exceptional object (1-dimensional vector space) and all triangulated categories generated by unique exceptional object are equivalent to this one. The basic examples of higher-dimensional minifolds are the projective spaces: by a theorem of Beilinson [1] projective d-dimensional space $\mathbb{P}^d$ has a full exceptional collection of line bundles $\mathcal{O},\mathcal{O}(1),\ldots,\mathcal{O}(d)$. Theorems of Kapranov, Orlov and Kuznetsov provide some other examples: odd-dimensional quadrics [12], threefolds V_5 [21] and V_{22} [14], the G_2-Grassmannian G_2/P and fivefold linear sections of the Lagrangian Grassmannian $LGr(3,6)$ or the spinor variety $OGr(5,10)$ [15].

Bondal and Orlov (circa 1995) conjectured that all minifolds are odd-dimensional rational Fano manifolds with a single exception of projective spaces. In theorem 1.1 we prove it in case of Fano fourfolds. These result can be considered as yet another homological characterization of the projective fourspace after theorems of Hirzebruch–Kodaira [11], Kobayashi–Ochiai [13], Fujita [8], Libgober–Wood [18], Bondal–Orlov [4].

Theorem 1.1. *Any fourfold X with a full exceptional collection of five coherent sheaves is the projective space.*

Proof. By [5] we may assume that X is a Fano fourfold with five-dimensional cohomology. Then by [23] X is either $\mathbb{P}^4$ itself, or a *Wilson's fourfold* [1] : a four-dimensional projective manifold with five-dimensional cohomology and Hilbert polynomial equal to $\chi(\omega_X^{-l}) = 1+\frac{25}{8}l(l+1)(3l^2+3l+2)$. It remains to prove the following lemma:

Lemma 1.2. *Wilson's fourfolds X do not admit a full exceptional collection in $\mathcal{D}^b_{coh}(X)$.*

□ In Section 2 we collect the necessary background on manifolds with full exceptional collections and minifolds; in Section 3 we introduce a method to prove that some manifold is not a minifold based only on knowledge of its Hilbert polynomial; in Section 4 we finish the proof of Lemma 1.2, the reduction to field of two elements serves as the main technical simplification; in the concluding Section 5 by Theorems 5.2,5.4 we explain that reduction to field of two elements in Section 4 was not a coincidence, namely any non-degenerate non-skew-symmetric bilinear form over any fields of more than two elements has a semi-orthogonal base, finally we formulate an open problem 5.6 of bilinear algebra, whose solution will lead to numerical classification of minifolds.

Very different situation of topologically minimal manifolds with ample canonical bundle is discussed in the next issue [9]. Also in the upcoming work [10] we'll show that projectivity assumption is superfluous, as well as prove some more general statements in arbitrary dimension.

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[1]A posteriori it turns out that all fourfolds X with the same Hodge numbers as $\mathbb{P}^4$ also have the same Pontrjagin numbers as $\mathbb{P}^4$, that is $p_1^2[X]=25$ and $p_2[X]=10$. We don't know of any conceptual explanation to this phenomenon: L-genus equals 1 by Hirzebruch index theorem, but why $\hat{A}$-genus should be the same as for $\mathbb{P}^4$ is a mystery.

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2. Preliminaries.

Definition 2.1. We say that an object E of a triangulated category $\mathcal{T}$ is *exceptional* if all spaces $\mathrm{Hom}(E, E[k])$ vanish except for 1-dimensional space $\mathrm{Hom}(E, E)$. Sequence of exceptional objects $E_1, \dots, E_R$ is an *exceptional collection* if spaces $\mathrm{Hom}(E_a, E_b[k])$ vanish for all $a > b$. If also $\mathrm{Hom}(E_a, E_b[k])$ vanish for all a, b and $k \neq 0$ it is called *strictly exceptional*. Exceptional collection is called *full* if minimal triangulated subcategory of $\mathcal{T}$ that contains $E_1, \dots, E_R$ coincides with $\mathcal{T}$. In this case we write $\mathcal{T} = \langle E_1, \dots, E_R\rangle$. We say that variety X has a full (strictly) exceptional collection if its bounded derived category of coherent sheaves $\mathcal{D}^b_{coh}(X)$ has one.

Proposition 2.2 (see [17])**.** *Hochschild homology $HH_i(X)$ of a projective variety X with a full exceptional collection of R objects in $\mathcal{D}^b_{coh}(X)$ vanish for $i \neq 0$ and $HH_0(X)$ is the R-dimensional vector space freely generated by Chern characters $ch(E_k)$.*

Proof. Hochschild homology is additive with respect to semiorthogonal decompositions, so $HH_k(\mathcal{D}^b_{coh}(X)) = \oplus_{a=1}^R HH_k(\langle E_a\rangle)$ where $\langle E_a\rangle$ is a triangulated category generated by an exceptional object E_a. The category $\langle E_a\rangle$ is equivalent to the derived category of coherent sheaves on a point i.e. the derived category of vector spaces, so $HH_k(\langle E_a\rangle) = 0$ for $k \neq 0$ and $HH_0(\langle E_a\rangle) = \mathbb{C}ch(E_a)$. □

Corollary 2.3. *Hodge numbers $h^{p,q}(X)$ vanish for $p \neq q$ and the total dimension of de Rham cohomology of X equals R. In particular odd Betti numbers $b_{2k+1}(X)$ vanish and Euler number of X equals R.*

Proof. Hochschild-Kostant-Rosenberg theorem implies $\dim HH_k(X) = \sum_{i-j=k} \dim H^{i,j}(X)$. Now the statement follows from proposition 2.2. □

Proposition 2.4. *Number R of objects in a full exceptional collection is bounded from below:*

$$R \geqslant (\dim X + 1).$$

If the bound is saturated then variety X has the same Hodge numbers as the equi-dimensional projective space.

Proof. Indeed, for any projective manifold $h^{p,p}(X) \geqslant 1$ for $0 \leqslant p \leqslant \dim X$. This observation and proposition 2.2 implies $R = \dim HH_0(X) = \sum_{p=0}^{\dim X} h^{p,p}(X) \geqslant (\dim X + 1)$. □

The minifolds were systematically studied by Bondal, Polishchuk and Positselski [2, 5, 22]. In particular they prove that if minifold X is not Fano then all full exceptional collections on it are not strict and consist not of pure sheaves:

Theorem 2.5 ([2, 5, 22])**.** *Assume $\langle E_1, \dots, E_{d+1}\rangle = \mathcal{D}^b_{coh}(X)$ is a full exceptional collection of $(d+1)$ objects on d-dimensional variety X. The following conditions are equivalent*

1. *the collection is strictly exceptional*
2. *for some $A \in \mathbb{Z}$ all objects $E_i[A]$ are vector bundles*
3. *for some $A \in \mathbb{Z}$ all objects $E_i[A]$ are sheaves*

Under any of these conditions X is a Fano variety, hence X is uniquely determined by $\mathcal{D}^b_{coh}(X)$ thanks to Bondal–Orlov:

Theorem 2.6 ([4])**.** *Let X and Y be two smooth projective varieties and assume that the anti-canonical line bundle of X is ample. If there exists an exact equivalence $\mathcal{D}^b_{coh}(X) \cong \mathcal{D}^b_{coh}(Y)$ then $X \cong Y$.*

3. Lattices and polynomials

Recall that K-theory $K_0(X) = K_0(\mathcal{D}^b_{coh}(X))$ is equipped with bilinear Euler pairing

$$\chi(E, F) = \sum_i (-1)^i \dim \mathrm{Hom}(E, F[i])$$

Proposition 3.1 (see [2, 5])**.** *K-theory of variety with a full exceptional collection of R objects is the free abelian group of rank R. It is freely generated by classes of objects $[E_k] \in K_0(X)$ and Euler pairing is non-degenerate and unimodular.*

Proof. The Gram matrix $e_{ab} = \chi(E_a, E_b)$ is uni-upper-triangular (i.e. base $[E_k]$ is semi-orthonormal). In particular e_{ab} is unimodular hence the Gram matrix is non-degenerate. So $[E_i]$ are linearly independent elements in $K_0(X)\otimes\mathbb{Q}$. Full exceptional collection generates the category, thus any object in $\mathcal{D}^b_{coh}(X)$ is equivalent to a linear combination of classes $[E_k] \in K_0(X)$. □

So far proposition 3.1 is the main bridge between the triangulated categories and the (bi)linear algebra. Some numerical properties of *Mukai lattices* $(K_0(X), \chi, e_\bullet)$ were investigated e.g. in [19, 20, 7, 6], but not as much is known as one would like to: most strong results are known only for latices of small rank (up to 4).

For any polynomial P of degree $\deg P \leqslant d$ set A^P to be $(d+1)\times(d+1)$-matrix

$$A^P_{i,j} = P(j-i) \tag{3.2}$$

Lemma 3.3. *The determinant of matrix A^P equals $(d!p_d)^{d+1}$, where $P(l) = p_d l^d + \dots$. In particular matrix A^P is non-degenerate $\iff$ $\deg P = d$.*

Proof. Let U, V be $(d+1)\times(d+1)$ matrices given by $U_{i,j} = (-1)^{i+j}\binom{i-1}{j-1}$, $V_{i,j} = (-1)^{i+j}\binom{d+1-j}{d+1-i}$. Note that U, V are lower-triangular with diagonal entries equal to 1. We consider UA^PV:

$$\begin{aligned}(UA^PV)_{i,j} &= \sum_{k,l=1}^{d+1}(-1)^{i+k+l+j}\binom{i-1}{k-1}P(l-k)\binom{d+1-j}{d+1-l} \\ &= (-1)^{i+j}\sum_{k,l=0}^{d}(-1)^{k+l}P(l-k)\binom{i-1}{k}\binom{d+1-j}{d-l} \\ &= (-1)^{i+j}\sum_{k,l=0}^{d}(-1)^{k+l+d}P(d-l-k)\binom{i-1}{k}\binom{d+1-j}{l}.\end{aligned} \tag{3.4}$$

Let T be the shift operator on the space of polynomials, $(Tf)(x) = f(x-1)$. We have

$$\begin{aligned}(UA^PV)_{i,j} &= (-1)^{i+j}\left(\sum_{k,l=0}^{d}(-1)^{k+l+d}\binom{j-1}{k}\binom{d+1-i}{l}T^{l+k}P\right)(d) \\ &= (-1)^{i+j+d}\left((1-T)^{d-i+j}P\right)(d).\end{aligned} \tag{3.5}$$

When $j > i$ we have $(1-T)^{d-i+j}P = 0$, hence UA^PV is lower-triangular. When $i = j$ we have $(1-T)^{d-i+j}P = d!p_d$, hence the diagonal entries of UA^PV are equal to $(-1)^d d!p_d$. Thus

$$\det A^P = \det(UA^PV) = \left((-1)^d d!p_d\right)^{d+1} = (d!p_d)^{d+1}.$$

□

Let Λ be the free lattice of rank $d+1$ with a base $e_0, \dots, e_d$. We write elements of Λ as column-vectors with integer entries with respect to the base $e_0, \dots, e_d$. Consider a homomorphism $\lambda : \Lambda \to K_0(X)$ that sends e_k to $[\mathcal{O}_X(k)]$, where $\mathcal{O}(1) = \det T_X$ is the anticanonical line bundle. Let A_X be the bilinear form of Euler pairing on $K_0(X)$ pulled back to Λ.

In base $e_0, \dots, e_d$ bilinear form A_X is given by $(d+1)\times(d+1)$ matrix A. By the Hirzebruch-Riemann-Roch theorem $A = A^{P_X}$ where $P_X(l) = \chi(\mathcal{O}_X(l))$ is a polynomial of degree d (the *Hilbert polynomial*) with top coefficient being the anticanonical degree $p_d = \frac{\int_{[X]} c_1(X)^d}{d!}$. So by lemma 3.3 pairing A_X is non-degenerate, thus map λ is injective.

Proposition 3.6. *If $\chi_{K_0(X)}$ has a semi-orthonormal base $B_1, \dots, B_{d+1}$ then bilinear form A_X has a semi-orthonormal base over rationals $\mathbb{Q}$, over the localized ring $\mathbb{Z}[\frac{1}{\det A_X}]$ and hence modulo any prime p that does not divide* $\det A_X$.

Proof. Map $\lambda : \Lambda \to K_0(X)$ is an isometry. Both A_X (by lemma 3.3) and $\chi_{K_0(X)}$ (by assumption) are non-degenerate pairings on lattices of the same rank $(d+1)$. This implies that $\lambda(\Lambda)$ is a full sublattice of finite index g in $K_0(X)$. By assumption $[\mathcal{O}(a)] = \sum G_{b,a}B_b$ for $0 \leqslant a \leqslant d$, G is an integer $(d+1)\times(d+1)$ matrix with $\det G = g \neq 0$, matrix $G^{-t}AG^{-1}$ is upper-unitriangular. Thus $\det A = g^2$ so G is invertible if $\det A_X$ is invertible. □

Proof of the last proposition 3.6 also implies

Proposition 3.7. *If either $\chi_{K_0(X)}$ or A_X has a semi-orthonormal base over some ring R then the determinant $\det A_X$ is a perfect square in R.*

Corollary 3.8. *The anticanonical degree $D = \int_{[X]} c_1(X)^d$ of any even-dimensional minifold X is a perfect square.*

Proof. If $\dim X$ is even then $(\dim X + 1)$ is odd so D is a perfect square $\iff$ $D^{\dim X+1}$ is a perfect square. By lemma 3.3 number $D^{\dim X+1} = \det A_X$ and by proposition 3.7 number $\det A_X$ is a perfect square. □ Last statement also follows from the perfectness of Poincare pairing on intermediate cohomology $H^{\dim X}(X, \mathbb{Z})$.

Remark 3.9. It is not known if Wilson's fourfolds (of either Fano or general type) are actually exist. In what follows we do not rule out the possibility of existence of Wilson's fourfolds X. Instead we prove that $K_0(X)$ does not admit any semi-orthonormal base. Then proposition 3.1 implies that $\mathcal{D}^b_{coh}(X)$ does not admit a full exceptional collection.

4. Proof of Lemma 1.2

Let X be any Wilson's fourfold.

Explicitly, we have

$$A = A_X = \begin{pmatrix} 1 & 51 & 376 & 1426 & 3876 \\ 1 & 1 & 51 & 376 & 1426 \\ 51 & 1 & 1 & 51 & 376 \\ 376 & 51 & 1 & 1 & 51 \\ 1426 & 376 & 51 & 1 & 1 \end{pmatrix}.$$

For $u, v \in \Lambda$ written as vectors we have $\chi(u, v) = u^t A v$.

Assume a full exceptional collection $E_1, \ldots, E_5$ exists.

By lemma 3.3 the determinant of matrix A equals $225^5 = 15^{10}$, in particular it is an odd number and by proposition 3.6 A hase a semi-orthonormal base over field $\mathbb{F}_2$ of two elements. We prove that this is not possible, i.e.

Proposition 4.1. *Let A_2 be the 5×5 matrix over $\mathbb{F}_2$ obtained by reduction of A modulo 2 and $(u, v) \mapsto u^t A_2 v$ be the corresponding bilinear form on vector space $V = \mathbb{F}_2^5$. There is no base e_1, e_2, e_3, e_4, e_5 of V such that $(e_i, e_j) = 0$ for $i > j$ and $(e_i, e_i) = 1$.*

Proof. Explicitly, we have

$$A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

We begin by making a few remarks.

(1) The automorphism $S_2 := A_2^{-1} A_2^{t}$ [2] satisfies $(u, v) = (v, S_2 u)$ for all u, v, so it preserves A_2, i.e. $(u, v) = (Su, Sv)$, equivalently $S_2^t A_2 S_2 = A_2$. We have

$$S_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and a direct computation shows that S_2 has order 8.

(2) There are precisely 12 vectors x such that $(x, x) = 1$ [3], they form two orbits under the action of S_2. One orbit of length 8 is generated by $a_1 := (1, 0, 0, 0, 0)^t$, another orbit of length 4 is generated by $b_1 := (1, 0, 1, 0, 0)^t$.

(3) If a base $e_1, e_2, \ldots, e_5$ is semi-orthonormal, i.e. satisfies the condition from the statement of the proposition, then for each i ($1 \leqslant i \leqslant 4$) the base obtained by replacing e_i, e_{i+1} with $e_{i+1}, e_i + e_{i+1}(e_i, e_{i+1})$ is also semi-orthonormal[4]

[2] Induced by Serre functor $\mathcal{S}_X = \otimes \omega_X[\dim X]$ (see [3, 6]).

[3] Indeed $(x, x) = 1 \iff$ point x does not lie on quadric $Q = \{x | (x, x) = 0\}$. The quadric Q has a unique singular point in $\mathbb{P}(V)$ so it has 19 points over $\mathbb{F}_2$ and its complement has 12 points.

[4] This transformation also lifts to the action of the braid group on the set of exceptional collections [3, 6].

Denote $a_i = S_2^{i-1} a_1$, $b_i = S_2^{i-1} b_1$ and $c = (a_1, \dots, a_8, b_1, \dots, b_4)$. The following matrix has (c_i, c_j) on position i, j:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Now we begin the proof. Assume there is a semi-orthonormal base. Then all of its vectors must be from the set $\{a_i\} \cup \{b_i\}$. Since there are only 4 vectors in $\{b_i\}$, at least one of the base vectors must be from $\{a_i\}$. Applying S_2 if necessary we may assume that this vector is a_1. Applying the transformation (3) we can obtain a semi-orthonormal base with a_1 on the first position.

Any remaining base vector x must satisfy $(x, a_1) = 0$. Looking at the first column of the matrix of (c_i, c_j) we see that the remaining base vectors must be from the set $\{a_3, a_4, a_5, a_6, b_1, b_2\}$. Let x be the second base vector. Then any vector y out of the remaining 3 base vectors must satisfy $(y, x) = 0$. However, trying for x each of the $\{a_3, a_4, a_5, a_6, b_1, b_2\}$ we see that there are only 2 choices remaining for y. This is a contradiction. □

5. New open problems in bilinear algebra

Our simple proof of theorem 1.1 looks a little bit surprising in view of the theorem 5.1, very surprising in view of the theorem 5.2 and is simply a needle-in-a-haystack in view of the theorem 5.4.

Theorem 5.1 ([6](Theorem 3.7))**.** *Non-degenerate bilinear form over complex numbers has a semi-orthonormal base* $\iff$ *it is not skew-symmetric.*

In fact, over any *infinite* field non-degenerate non-skew-symmetric bilinear form admits a semi-orthogonal base (similarly to well-known statement that every quadratic form admits an orthogonal base). This result implies Theorem 5.1. We say that a bilinear form is non-skew-symmetric if the corresponding quadratic form is non-zero. This definition reduces to the usual one if the base field has characteristic different from 2.

Theorem 5.2. *A non-degenerate non-skew-symmetric bilinear form over an infinite field has a semi-orthogonal base.*

Proof. Let k be an infinite field. We make use of the following fact.

Lemma 5.3. *A non-empty Zariski-open subset of an affine space over* k *has a* k*-rational point.*

Let V be an n-dimensional vector space over k with a non-degenerate non-skew-symmetric bilinear form $(\cdot, \cdot)$. The proof goes by induction on n.

Let $x \in V$ be a vector with $(x, x) \neq 0$. Such a vector exists by Lemma 5.3.

Consider the affine space $X = V^{n-1}$. Let X_1 be the Zariski-open subset of X defined by

$$X_1 = \{(e_1, \dots, e_{n-1}) \in X \mid \text{the matrix } (e_i, e_j) \text{ is non-skew-symmetric.}\}$$

The algebraic set X_1 is not empty because it contains $(x, 0, \dots, 0)$.

Let X_2 be the Zariski-open subset of X determined by

$$X_2 = \{(e_1, \dots, e_{n-1}) \in X \mid \det(e_i, e_j) \neq 0\}.$$

The algebraic set X_2 is not empty because it contains any base of $x^\perp = \{y \in V | (x, y) = 0\}$. Indeed, let $e_1, \dots, e_{n-1}$ be a base of $x^\perp$. Put $e_n = x$. Then $e_1, \dots, e_n$ is a base of V. We have

$$0 \neq \det(e_i, e_j)_{i,j=1}^n = (e_n, e_n) \det(e_i, e_j)_{i,j=1}^{n-1}.$$

Therefore $(e_1, \dots, e_{n-1}) \in X_2$.

By Lemma 5.3 we can choose $(e_1, \ldots, e_{n-1}) \in X_1 \cap X_2$. Let $U \subset V$ be the linear span of $e_1, \ldots, e_{n-1}$. Then the restriction of $(\cdot, \cdot)$ to U is non-degenerate and non-skew-symmetric. By the induction hypothesis we can choose a semi-orthogonal base $v_2, \cdots, v_n$ of U. Let v_1 be a non-zero vector in $U^\perp = \{y \in V | (\forall u \in U)(u, y) = 0\}$. Then $v_1 \notin U$ because the restriction of $(\cdot, \cdot)$ to U is non-degenerate. Therefore $v_1, \cdots, v_n$ is a semi-orthogonal base of V. □

Theorem 5.4. *A non-degenerate non-skew-symmetric bilinear form over a finite field different from $\mathbb{F}_2$ has a semi-orthogonal base.*

Proof. The proof goes similarly to the proof of Theorem 5.2, but instead of Lemma 5.3 we use point counting. Let the base field be $\mathbb{F}_q$, the vector space V and the bilinear form $(\cdot, \cdot)$. Denote the corresponding quadratic form by Q, $Q(x) = (x, x)$.

First we bound the number $\#\{x \in V : Q(x) = 0\}$. We can find a vector $v \in V$ with $Q(v) \neq 0$. Indeed, write $Q = \sum_{i \leqslant j} Q_{ij} x_i x_j$ in some basis. If for some i the coefficient $Q_{ii} \neq 0$, then the vector v with $v_j = \delta_{ij}$ satisfies $Q(v) = Q_{ii} \neq 0$. Otherwise for some i, j we have $Q_{ij} \neq 0$. Then the vector v with $v_k = \delta_{ik} + \delta_{jk}$ satisfies $Q(v) = Q_{ij} \neq 0$. We complete v to a basis of V. In this basis Q has the form $Q = ax_1^2 + b(x_2, \ldots, x_n)x_1 + c(x_2, \ldots, x_n)$ with $a \neq 0$. Then for each choice of $x_2, \ldots, x_n$ we have at most 2 values of x_1 with $Q(x) = 0$. For the choice $x_2 = \cdots = x_n = 0$ we have only one value of x_1 with $Q(x) = 0$. Thus

$$\#\{x \in V : Q(x) = 0\} \leqslant 2q^{n-1} - 1.$$

The number of lines on which the restriction of Q is zero is at most $2\frac{q^{n-1}-1}{q-1}$.

Second we bound the number of hyperplanes on which the restriction of Q is trivial. Through each of the lines with trivial restriction of Q we can draw $\frac{q^{n-1}-1}{q-1}$ hyperplanes. All the hyperplanes with trivial restriction of Q are among these, and each contains $\frac{q^{n-1}-1}{q-1}$ lines with trivial restriction of Q. Thus we have at most $2\frac{q^{n-1}-1}{q-1}$ hyperplanes with trivial restriction of Q.

The number of lines l such that $Q|_l \neq 0$ and $Q|_{l^\perp} \neq 0$ is thus at least

$$\frac{q^n - 1}{q - 1} - 4\frac{q^{n-1} - 1}{q - 1} = \frac{q^n - 4q^{n-1} + 3}{q - 1}.$$

If $q \geqslant 4$, this is positive, so there exists such l. The restriction of $(\cdot, \cdot)$ to $l^\perp$ is non-degenerate and non-skew-symmetric and we can proceed as in the proof of Theorem 5.2.

It remains to consider the case $q = 3$. In fact the following proof is valid for any odd q. The case $n = 2$ can be treated separately by choosing any vector v_1 with $Q(v_1) \neq 0$ and a non-zero v_2 with $(v_2, v_1) = 0$. Suppose $n \geqslant 3$. Let $\omega(\cdot, \cdot)$ be the symmetric bilinear form such that $Q(x) = \omega(x, x)$. Such form exists because q is odd. If there exists a hyperplane with trivial restriction of Q then it is not hard to show that $\dim \operatorname{Ker} \omega$ is $n - 2$ or $n - 1$. If $\dim \operatorname{Ker} \omega = n - 1$ there is only one such hyperplane, namely $\operatorname{Ker} \omega$. If $\dim \operatorname{Ker} \omega = n - 2$, such hyperplanes correspond to lines in $V/\operatorname{Ker} \omega$ on which the restriction of the form induced from ω is trivial. There are at most 2 such lines because $\dim(V/\operatorname{Ker} \omega) = 2$.

Thus in the case q odd and $n \geqslant 3$ the number of lines l such that $Q|_l \neq 0$ and $Q|_{l^\perp} \neq 0$ is at least

$$\frac{q^n - 1}{q - 1} - 2\frac{q^{n-1} - 1}{q - 1} - 2 = \frac{q^n - 2q^{n-1} - 2q + 3}{q - 1} \geqslant \frac{q^n - 3q^{n-1} + 3}{q - 1}.$$

When $q \geqslant 3$ this is positive and we proceed in the same way as before. □

Definition 5.5. We say that $P(l)$ is an *FHM*[5] *polynomial* if it satisfies the following properties:

(1) $P(\mathbb{Z}) \subset \mathbb{Z}$ (dimension of cohomology is integer)
(2) $P(-1 - l) = (-1)^d P(l)$ (Serre duality)
(3) $P(0) = q_0 = 1$ (Todd genus equals 1)

Note that the first condition $P(\mathbb{Z}) \subset \mathbb{Z}$ is equivalent to $P(l) = \sum_{j=0}^{d} q_j \binom{l}{j}$ with $q_j \in \mathbb{Z}$.

It is clear that if $P(l) = \chi(\omega_X^{-l})$ then the first two conditions hold for any Gorenstein variety X and the third condition holds for minifolds and Fano manifolds.

Our proof suggests the following problem of (linear) algebra.

[5] After Fano, Hilbert and Macaulay.

Problem 5.6. *Classify FHM polynomials $P(l) = \sum_{k=0}^{d} q_k \binom{l}{k}$ such that the bilinear form given by matrix A^P has a semi-orthonormal base over the localized ring $\mathbb{Z}[\frac{1}{\det A^P}]$.*

Note that by Lemma 3.3 we have $\det A^P = q_d^{d+1}$, so $q_d \neq 0$. By Proposition 3.7 if d is even then q_d should be a perfect square. Under these assumptions by Theorem 5.4 bilinear form A^P has a semi-orthonormal base over any finite field of $q > 2$ elements.

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