SMALL TORIC DEGENERATIONS OF FANO 3-FOLDS

SERGEY GALKIN

Abstract. We show which of the smooth Fano 3-folds admit degenerations to toric Fano 3-folds with ordinary double points.

1. Introduction

We consider small toric degenerations of Fano 3-folds, that are degenerations of smooth Fano 3-folds to toric Fano 3-folds with ordinary double points (see definitions 2.1, 2.2). That kind of degenerations has applications in mirror symmetry. Mirror symmetry for smooth toric varieties (and complete intersections in such varieties) was constructed by Givental and Batyrev in [1], [2] and [3].

If smooth Fano Y admits a small toric degeneration X, one can construct a candidate for mirror of Y via toric construction, and hence compute some Gromov–Witten invariants of Y.

In that way, using small toric degenerations of Grassmannians (constructed in [4]) and varieties of partial flags (constructed in [5]), the candidates for mirrors of these homogenuous varieties were constructed in [6], [7].

Generalizing these examples Batyrev introduced the notion of small toric degeneration of a Fano variety in [8]. The complete classification of smooth Fano 3-folds is well known due to works of Iskovskikh, Shokurov, Mori and Mukai ([9],[11],[10],[12], see also [13],[14] and [15]).

So, Batyrev posed a natural question ([8, Question 3.9]): «Which 3-dimensional nontoric smooth Fano varieties do admit small toric degenerations?»

Theorem 2.7 of this paper provides an answer. Section 6 contains an example application of these degenerations.

2. The claim

Definition 2.1. Deformation is a flat proper morphism

$$\pi : \mathcal{X} \to \Delta,$$

where Δ is a unit disc $\{|t| < 1\}$, and $\mathcal{X}$ is an irreducible complex manifold. All the deformations we consider are projective (π is a projective morphism over Δ). Denote fibers of π by X_t, and let $i_{t\in\Delta}$ be the inclusion of a fiber $X_t \to X$.

If all fibers $X_{t\neq 0}$ are nonsingular, then the deformation π is called a degeneration of $X_{t\neq 0}$ or a smoothing of X_0. If at least one such morphism π exists, we say that varieties $X_{t\neq 0}$ are *smoothings* of X_0, and X_0 is a *degeneration* of $X_{t\neq 0}$.

For a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ over Δ and $t \in \Delta$ the symbol $\mathcal{F}_t$ stands for the restriction $i_t^*\mathcal{F}$ to the fiber over t. In particular there is a well-defined restriction morphism on Picard groups $i_t^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_t)$.

Definition 2.2. [[8]] Degeneration (or a smoothing) π is *small*, if X_0 has at most Gorenstein terminal singularities (see [16] or [17]), and for all $t \in \Delta$ the restriction $i_t^* : \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_t)$ is an isomorphism.

All 3-dimensional terminal Gorenstein toric singularities are nodes i.e. ordinary double points analytically isomorphic to $(xy = zt) \subset \mathbb{A}^4$ (see e.g. [18]).

Definition 2.3. The *index* of a (Gorenstein) Fano variety X is the highest $r > 0$, s.t. anticanonical divisor class $-K_X$ is an r-multiple of some integer Cartier divisor class H:

$$-K_X = rH.$$

Definition 2.4. Let $H \in \operatorname{Pic}(X)$ be a Cartier divisor an on n-dimensional variety X, and $D_1, \ldots, D_l$ be a base of lattice $H^{2k}(X, \mathbb{Z})/\text{tors}$. Define $d^k(X, H)$ as a discriminant of the quadratic form $(D_1, D_2) = (H^{n-2k} \cup D_1 \cup D_2)$ on $H^{2k}(X, \mathbb{Z})/\text{tors}$. For a Gorenstein threefold X denote by $d(X) = d^1(X, -K_X)$ the anticanonical discriminant of X.

If X is a smooth variety and H is an ample divisor, then hard Lefschetz theorem states that $d^k(X, H)$ is nonzero.

Definition 2.5. Let X be a Fano threefold. Consider Picard number $\rho = \operatorname{rk} \operatorname{Pic}(X) = \dim H^2(X)$, half of third Betti number $b = \frac{1}{2} \dim H^3(X)$, (anticanonical) degree $\deg = (-K_X)^3$, Fano index r (see def. 2.3) and (anticanonical) discriminant d (see def. 2.4). Numbers ρ, r, deg, b, d form a set of *principal invariants* of smooth Fano 3-fold.

Definition 2.6. We use the following notations for the (families of) smooth varieties

- $\mathbb{P}^n$ — n-dimensional projective space;
- Q_n — n-dimensional quadric in $\mathbb{P}^{n+1}$;
- $G(l, N)$ — Grassmanian of l-dimensional linear subspaces in in N-dimensional space.

  families of smooth surfaces:
- $\mathbb{F}_n$, $n > 0$ — rational ruled surface (Hirzebruch surface) $\mathbb{P}_{\mathbb{P}^1}(\emptyset \oplus \emptyset(n))$; $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, F_1 is a blowup of $\mathbb{P}^2$ in a point;
- S_d, $d = 1, \ldots, 8$ — del Pezzo surfaces of degree d and index 1 ($S_8 = \mathbb{F}_1$);

  and families of smooth Fano threefolds:
- $Q = Q_3$ — a quadric in $\mathbb{P}^4$;
- V_4 — intersections of two quadrics in $\mathbb{P}^5$;
- $V5$ — a section of $G(2, 5)$ by linear subspace of codimension 3;
- V_{22} — Fano threefolds of genus 12 with $\rho = 1$;
- W — divisor of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (i.e. $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$);
- V_7 — a blowup of $\mathbb{P}^3$ in a point (i.e. $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$);
- $V_{\rho.N}$ ($\rho = 2, 3, 4$) — families of Fano threefolds with Picard number ρ and number N in Mori–Mukai's tables [10, Table 2, Table 3, Table 4].

We use the standart notations for toric varieties ([19], [20], [1]): a toric variety X corresponding to a fan Σ in the space $N = \mathbb{Z}^{\dim X}$, each ample divisor H on X corresponds to a polytope Δ_H in the dual space $M = \operatorname{Hom}(N, \mathbb{Z})$; we denote the variety X by the symbols X_Σ or $\mathbb{P}(\Delta)$.

Theorem 2.7. *These and only these families of nontoric smooth Fano 3-folds Y do admit small degenerations to toric Fano threefolds (in notations 2.6):*

(1) *4 families with* $\mathrm{Pic}(Y)=\mathbb{Z}$*:* $Q,\ V_4,\ V_5,\ V_{22}$*;*
(2) *16 families with* $\mathrm{Pic}(Y)=\mathbb{Z}^2$*:* $V_{2.n}$*, where* $n = 12,\ 17,\ 19,\ 20,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27,\ 28,\ 29,\ 30,\ 31,\ 32$*;*
(3) *16 families with* $Pic(Y)=\mathbb{Z}^3$*:* $V_{3.n}$*, where* $n = 7,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20,\ 21,\ 22,\ 23,\ 24$*;*
(4) *8 families with* $Pic(Y)=\mathbb{Z}^4$*:* $V_{4.n}$*, where* $n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$*.*

All these degenerations are listed in section 5.

Remark 2.8. Aposteriori all these smooth 3-folds Y satisfy the following conditions
(1) Y is rational (see e.g. [15]),
(2) $\rho(Y) \leq 4$,
(3) $\deg(Y) = (-K_Y)^3 \geq 20$,
(4) $b(Y) = h^{1,2}(Y) \leq 3$,
(5) $b(Y) = 3$ only if Y is $V_{2.12}$,
(6) $b(Y) = 2$ only if Y is V_4 or $V_{2.19}$.

3. The proof

A sketch of the proof. Consider a toric Fano 3-fold X with ordinary double points.

(i): There is only a finite number of such X. All these threefolds X are explicitly classified.
(ii): X admits a smoothing — a Fano threefold Y ([27]).
(iii): Principal invariants of Y can be expressed via invariants of X.
(iv): Family of smooth Fano 3-folds Y is completely determined by its principal invariants.
(v): If some smooth Fano threefold Y admits a degeneration to a nodal toric Fano X, then the pair (Y, X) comes from the steps (i)-(iv).

The following properties of Fano varieties are consequences of Kawamata–Viehweg theorem, exponential sequence and Leray spectral sequence.

Proposition 3.1. *(See e.g.* [15], [16]*) Let X be an almost Fano with canonical singularities. Then*
(1) $H^i(X, \mathcal{O}) = 0$ *for all i>0,*
(2) $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$*,*
(3) $\mathrm{Pic}(X)$ *is a finitely generated free* $\mathbb{Z}$*-module.*

If $\pi : Y \to X$ *is a resolution of singularities, the listed properties hold also for* Y *, and* $R^{\text{ffl}}\pi_*\mathcal{O}_Y = \mathcal{O}_X$ *(i.e. canonical singularities are rational).*

Local topology of smoothings is described by the following

Proposition 3.2. *(see e.g.* [21], [22] *or* [23]*) Let* $\pi : \mathcal{X} \to \Delta$ *be a smoothing.*
(1) *Restriction* $\pi : \mathcal{X}\backslash\mathcal{X}_0$ *is a locally trivial fibration of smooth topological manifolds, in particular all the smooth fibers are diffeomorphic (this is known as Ehresmann's theorem).*
(2) *There is a continuous Clemens map* $c : \mathcal{X} \to X_0$ *(outside* $c^{-1}(\mathrm{Sing}\, X_0)$*) the map* c *is smooth). Clemens map* c *is a deformation retraction of* $\mathcal{X}$ *to the fibre* X_0 *and respects the radial retraction* $\Delta \to 0$*. Restriction of* c *to the smooth fiber* X_t *is 1-to-1 correspondence outside singular locus of* X_0*.*

These propositions are purely topological, and essentialy are the variations of the tubular neighborhood theorem.

Corollary 3.3. *$\mathcal{X}$ and X_0 has the same homotopy type (the homotopy equivalences are given by the Clemens map $c : \mathcal{X} \to X_0$ and the inclusion of the fiber $i_0 : X_0 \to \mathcal{X}$). Hence*

$$H^2(X_0, \mathbb{Z}) = H^2(\mathcal{X}, \mathbb{Z}),$$
$$H_2(X_0, \mathbb{Z}) = H_2(\mathcal{X}, \mathbb{Z}).$$

Corollary 3.4. *For $t \neq 0$ all the images* $\mathrm{Im}[\{i_t\}_{:}H_\bullet(X_t, \mathbb{Z}) \to H_\bullet(\mathcal{X}, \mathbb{Z})]$ *coincide.*

Proof. Let U_i be the covering of $\Delta \backslash 0$ s.t. π is locally trivial fibration over elements of the covering U_i. Consider a pair of points $t,\ s \in U_i$ and a k-cycle $\gamma \in H_k(X_t, \mathbb{Z})$. Let $I \subset U$ be an interval between t and s in U, and γ_U be a $(k+1)$-cycle in $\mathcal{X}_I$, corresponding to the product of I and γ in a fixed trivialization of π over I. Then the boundary of γ_U in $\mathcal{X}$ is equal to the difference between $\{i_t\}_*\gamma$ and $\{i_s\}_*\gamma$. □

Theorem 3.5 ([24])**.** *Hodge numbers $h^{p,q}(X_t)$ are constant for all $t \in \Delta \backslash 0$.*

Proposition 3.6 (Semicontinuity theorem, see e.g. [25])**.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, flat over $\mathcal{O}_\Delta$; put $\mathcal{F}_t = i_t^*$. Then*

(1) *The Euler characteristic $\chi(X_t, \mathcal{F}_t)$ does not depend on $t \in \Delta$.*
(2) *Dimension of $H^i(X_t, \mathcal{F}_t)$ is upper-semicontinuous as a function of t (i.e. for all $n \in \mathbb{Z}$ sets)*

$$\{t \in \Delta : h^i(X_t, \mathcal{F}_t) \geq n\}$$

are closed in Zariski topology).

Remark 3.7. We will use the following trick: if the cohomologies of some coherent sheaf $H^i(X_0, \mathcal{F})$ vanish, then assume that Δ is chosen small enough, s.t. vanishing holds for the cohomologies of all the fibers over Δ.

Theorem 3.8 ([26])**.** *Let X_0 be a variety with canonical singularities, and $\mathcal{X}$ — a deformation. Then total space $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein (Gorenstein if X_0 is) and admit only canonical singularities.*

In this case one can use the naive adjunction formula on X (dualizing sheaf coincides with the canonical one).

Assume that X_0 is Gorenstein and admits at most canonical singularities, and either a Calabi-Yau of dimension ≥ 2 or almost Fano.

Proposition 3.9. *For all i and t*

$$h^i(X_t, \mathcal{O}_{X_t}) = h^i(X_0, \mathcal{O}_{X_0}).$$

Proof. Consider $h^i(X_t, \mathcal{O}_{X_t})_{0<i<\dim X_t}$ as a function of t. It is upper-semicontinuous (see 3.6 (ii)), and equal to 0 for $t = 0$ (by the definition if X is Calabi-Yau, or by 3.1 if X is almost Fano). Hence this function is 0 in some neighborhood of 0. This means it is identical to 0 over Δ (Theorem 3.5). Since $h^0(X_t, \mathcal{O}) = 1$ for all t, from Proposition 3.6 (i) if follows that $h^n(X_t) = h^n(X_0)$ for all t (it is equal to 0 in case of almost Fano and 1 for Calabi-Yau). □

By Grauerth's theorem $R^i\pi_*\mathcal{O} = R^i\pi_*\mathcal{O}(-K_{\mathcal{X}})$, $\dim X_0 > i > 0$, and $\pi_*\mathcal{O}(-K_{\mathcal{X}})$ is a locally free sheaf over Δ of rank $h^0(X_0, \mathcal{O}(-K_{X_0}))$. From the degenerations of Leray spectral sequences

$H^i(\Delta, R^j\pi_*\mathcal{O}(-K_{\mathcal{X}}))$ and $H^i(\Delta, R^j\pi_*\mathcal{O})$:

$$H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-K_{\mathcal{X}})) = H^i(X_t, \mathcal{O}_{X_t}(-K_t)) = 0,\ \dim X_0 > i > 0,\ t \in \Delta \tag{3.10}$$

$$H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^i(X_t, \mathcal{O}_{X_t}) = 0,\ \dim X_0 > i > 0,\ t \in \Delta \tag{3.11}$$

$$H^0(X_t, \mathcal{O}_{X_t}(-K_{X_t})) = H^0(X_0, \mathcal{O}_{X_0}(-K_{X_0})),\ t \in \Delta \tag{3.12}$$

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-K_{\mathcal{X}})) = H^0(X_0, \mathcal{O}_{X_0}(-K_{X_0})) \otimes H^0(\Delta, \mathcal{O}). \tag{3.13}$$

By exponential sequence and the vanishing 3.11 there are isomorphisms

$$\mathrm{Pic}(\mathcal{X}) = H^2(\mathcal{X}, \mathbb{Z}), \tag{3.14}$$

$$\mathrm{Pic}(X_t) = H^2(X_t, \mathbb{Z}) \tag{3.15}$$

Next proposition is a combination of 3.14 and 3.3:

Proposition 3.16. $i_0^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_0)$ *is an isomorphism.*

Proposition 3.17. $i_t^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_0)$ *is injective, i.e.*

$$\mathrm{Ker}\, i_t^* = 0. \tag{3.18}$$

Proof. Since for all $\gamma \in H_\bullet(X_t)$ and $\Gamma \in H^\bullet(\mathcal{X})$ we have

$$\langle i_t^*(\Gamma), \gamma\rangle = \langle \Gamma, \{i_t\}_*\gamma\rangle,$$

so from nondegeneracy of the coupling on X_t for $t \neq 0$ and Corollary 3.4 we conclude that the spaces $\mathrm{Ker}\, i_t^* : H^2(\mathcal{X}, \mathbb{Z})\backslash \mathrm{tors} \to H^2(X_t, \mathbb{Z})$ coincide for all $t \neq 0$. Isomorphism 3.14 implies the same holds for $i_t : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_t)$, i.e. $\mathrm{Ker}\, i_t = \mathrm{Ker}\, i_{t'}$ for all $t, t' \in \Delta\backslash 0$.

Consider an element $\mathcal{L} \in \mathrm{Ker}\, i_t^* = \cap_{t' \in \Delta\backslash 0} \mathrm{Ker}\, i_{t'}^*$. Then $\mathcal{L}$ is inversible sheaf with the property $\mathcal{L}_{X_t} = \mathcal{O}_{X_t}$, $t \in \Delta\backslash 0$. If $t \neq 0$ this trivial line bundle has 1-dimensional space of sections:

$$h^0(X_t, \mathcal{L}_{X_t}) = h^0(X_t, \mathcal{O}_{X_t}) = 1,$$

so by semicontinuity (Proposition 3.6)

$$h^0(X_0, \mathcal{L}_{X_0}) \geq 1.$$

In the same way

$$h^0(X_0, \mathcal{L}_{X_0}^{-1}) \geq 1.$$

This means $\mathcal{L}_{X_0} \cong \mathcal{O}_{X_0}$. So 3.16 implies $\mathcal{L} \cong \mathcal{O}_{\mathcal{X}}$. $\square$

By 3.8 and adjunction formula for all $t \in \Delta$

$$-K_{X_t} = -(K_{\mathcal{X}} + X_t)|_{X_t} = i_t^*(-K_{\mathcal{X}}). \tag{3.19}$$

Consider $\mathcal{D} \in \mathrm{Pic}(\mathcal{X})$. The fibres $X_0 = X$ and X_t are algebraically equivalent, so

$$i_0^*(\mathcal{D})^{\dim X} = \mathcal{D}^{\dim X} \cdot X_0 = \mathcal{D}^{\dim X} \cdot X_t = y_t^*(\mathcal{D})^{\dim X}. \tag{3.20}$$

Corollary 3.21. *Anticanonical degree* $(-K_{X_t})^{\dim X_t}$ *does not depend on* $t \in \Delta$.

Let $\mathcal{X}$ be a relative Fano (i.e. $-K_{\mathcal{X}}$ is ample over Δ).

Theorem 3.22 ([27])**.** *Any Fano 3-fold* X_0 *with ordinary double points admits a smoothing* $\pi : \mathcal{X} \to \Delta$ *with general fibers* $X_{t\neq 0}$ *being a smooth Fanos.*

Friedman's theorem 3.22 has a generalization to Gorenstein terminal singularities by Namikawa

Theorem 3.23 ([28]). *Any Gorenstein terminal Fano 3-fold X_0 admits a smoothing $\pi : \mathcal{X} \to \Delta$ with general fibers $X_{t\neq 0}$ being smooth Fanos.*

Proposition 3.24. *If X_0 is (almost) Fano, then the smoothing is small if and only if two pairs of invariants (ρ, d) coincide (d is defined in 2.4:*

$$\rho(X_0) = \rho(X_t),$$
$$d(X_0) = d(X_t).$$

Proof. Bijectivity of i_0^* and injectivity of i_t^* holds in general context (Proposition 3.17). Both groups $\mathrm{Pic}(X_t)$ and $\mathrm{Pic}(X)$ are finitly generated lattices (Proposition 3.1). Thus equality $\rho(X_0) = \rho(X_t)$ means that the morphism $i_t^*(i_0^*)^{-1}$ is an isomorphism of lattice $\mathrm{Pic}(X_0)$ with sublattice of finite index in $\mathrm{Pic}(X_t)$. This index is equal to

$$[\mathrm{Pic}(X_t) : \mathrm{Pic}(X_0)] = \left(\frac{d(X_0)}{d(X_t)}\right)^{1/2}.$$

□

Theorem 3.25 ([29]). *If $\mathcal{X}$ is a smoothing, and X_0 is a Gorenstein Fano 3-fold with terminal singularities, then i_t^* is an isomorphism for all t.*

Corollary 3.26. *Any Gorenstein Fano 3-fold with terminal singularities admits a smoothing, with general fiber being a smooth Fano 3-fold, and all such smoothings are small.*

Proof. This is just a union of 3.23 and 3.25. □

Corollary 3.27. *Gorenstein Fano 3-fold X with terminal singularities and its smoothing Y has the same invariants ρ, $\deg$, r, d.*

Proof. Equality 3.21 states that $\deg(X) = \deg(Y)$. As a corollary of 3.25 we have $\rho(X) = \rho(Y)$. Hence from 3.20 and 3.24 one derives $d(X) = d(Y)$. Finally, 3.25 with 3.19 implies $r(X) = r(Y)$. □

Fano 3-fold Y has only 2 non-trivial Hodge numbers $h^{1,1}(Y) = h^{2,2}(Y) = \rho(Y)$ and $b(Y) = h^{1,2}(Y) = h^{2,1}(Y) = \frac{1}{2}\operatorname{rk} H^3(Y,\mathbb{Z})$; and some trivial: $h^{0,0}(Y) = h^{3,3}(Y) = 1$, all other Hodge numbers are zeroes.

Proposition 3.28. *Let X be a nodal threefold, $\widetilde{X} \to X$ — its small crepant resolution, and Y — a smoothing of X (in literature transformation from Y to $\widetilde{X}$ is called a conifold transition). Denote the number of nodes on X by $p(X)$. Then*

$$b(Y) = p(X) + b(\widetilde{X}) + \rho(Y) - \rho(\widetilde{X}). \tag{3.29}$$

Proof. (*Clemens's argument, see also* [40]) Compare topological Euler numbers (for noncompact manifolds with a border use Euler number for cohomologies with compact support $\chi(M) = \sum_i (-1)^i \dim H_c^i(M,\mathbb{C})$) of $\widetilde{X}$ and Y[1].

By throwing away small neighborhoods of all singular points p_i from X, we construct a manifold with the border M. Punctured neighborhood of ordinary double point on X is isomorphic

[1]Alternatively one can compare dimensions of versal deformation spaces for Y and X; see also mirror-symmetry explanation [30], [6].

to tangent bundle on real sphere TS^3 without the 0-section: if $\sum_{i=1}^4 z_i^2 = 0$, $z = x + yi$ then x and y can be considered as a pair of nonzero orthogonal (w.r. to standart euclidean metric) vectors in $\mathbb{R}^4$ of the same length r; vector x/r is a point in $(n-1)$-dimensional sphere of radius 1, and y is a tangent vector in that point. This shows that a neighborhood of ordinary double point on X is isomorphic to $S^2 \times S^3$. After crepant resolution it is patched by $S^2 \times D^4$, and after smoothing — by $D^3 \times S^3$. Hence

$$\chi(\widetilde{X}) = \chi(M) + p \cdot \chi(S^2),$$
$$\chi(Y) = \chi(M) + p \cdot \chi(S^3).$$

This implies

$$\chi(\widetilde{X}) = \chi(Y) + 2p.$$

But

$$\chi(Y) = 2 + 2\rho(Y) - 2b(Y),$$
$$\chi(\widetilde{X}) = 2 + 2\rho(\widetilde{X}) - 2b(\widetilde{X}).$$

□

Proposition 3.30. *If X is a nodal toric threefold corresponding to a polytope with v vertices, p quadrangular faces (i.e. nodes) and $f-p$ triangular faces (smooth fixed points), then $H^3(\widetilde{X}) = 0$, $\rho(\widetilde{X}) = v - 3$. So for smoothing Y of X, there is a relation*

$$b(Y) = p + \rho(X) - (v-3).$$

Proof. Since $\widetilde{X}$ is nonsingular, $\mathrm{Pic}(\widetilde{X})$ and $\mathrm{Cl}(\widetilde{X})$ coincides. But the resolution $\widetilde{X} \to X$ is small, hence the proper transform is the bijection between Weyl divisors on $\widetilde{X}$ and X, i.e. $\mathrm{Cl}(\widetilde{X}) = \mathrm{Cl}(X)$. This implies $\rho(\widetilde{X}) = \mathrm{rk}\,\mathrm{Pic}(\widetilde{X}) = \mathrm{rk}\,\mathrm{Cl}(X) = v - 3$. Therefore proposition 3.28 in our case is equivalent to the equality 3.29. □

Theorem 3.31 ([10],[12]). *Two smooth Fano 3-folds Y_1, Y_2 with coincident sets of principal invariants ρ, r, $\deg$, b, d lie in one deformation class. There are only 105 such classes[2]. They are explicitly listed in* [10], *and nonempty.*

Let us say that smooth Fano threefold Y is determined by its invariants $(\rho,\ r,\ \deg,\ b)$, if for any smooth Fano threefold Y' equalities $\rho(Y') = \rho(Y),\ \rho(Y') = \rho(Y),\ \deg(Y') = \deg(Y),\ b(Y') = b(Y)$ imply that Y and Y' lie in one deformation class. According to [12], only 19 of 105 families of smooth Fano threefolds are not determined by invariants $\rho,\ r,\ \deg,\ b$.

Lemma 3.32. *For any nodal Fano threefold X there exists only one (up to deformations) smooth Fano Y, such that Y is a smoothing of X.*

Proof. X has a smoothing — a smooth Fano variety Y (see 3.22). Principal invariants of Y (see 2.5) are explicitly computable from invariants of X (see 3.27, 3.30). Deformation class of Y is uniquely determined by its principal invariants 3.31. □

[2]In the first version of [10] one family $V_{4.13}$ was missing, it was corrected in 2003.

Corollary 3.33. *Suppose Y is determined by (ρ, r, $\deg$, b). Then nodal Fano 3-fold X is a degeneration of Y if and only if $\rho(X) = \rho,\ r(X) = r,\ \deg(X) = \deg,\ b(X) = b$. If Y is not determined by (ρ, r, $\deg$, b), then X is a degeneration of Y if and only if $\rho(X) = \rho,\ r(X) = r,\ \deg(X) = \deg,\ b(X) = b,\ d(X) = d$.*

The proof of lemma 3.32 works in higher generality — not only in case of toric varieties, but for any nodal Fano threefolds (and also it is easy to generalize it to the case of Fano threefolds with Gorenstein terminal singularities). In the next part of the paper we restrict ourselves to the case of toric varieties X[3].

There is an effective algorithm describing all the reflexive polytopes (i.e. Gorenstein toric Fanos) in any fixed dimension (see [32]). Number of such polytopes grows fast enough: there are 16 polygons, 4319 polytopes in 3-dimensional space ([32]), and 473800776 4-imensional polytopes.

We are interested in the particular case of nodal toric Fano 3-folds. We used PALP software package ([31], [32]) to form a list of such varieties. There are 100 of them, 18 are smooth and aren't deformations of other smooth Fanos (theorem 3.31). For nonsmooth cases Picard number is not greater than 4. All these varieties are listed in the table of section 5[4].

So let us compute invariants of the smoothing Y of toric nodal Fano X.

Let $\pi : \widetilde{X} \to X$ be some small crepant resolution of X, and $p(X)$ be a number of nodes on X.

Proof of 2.7. Assume smooth Fano threefold Y is degenerated to X. As shown in 3.4, varieties X and Y have the same Picard number, index, anticanonical degree and invariant d. Denote them by

$$\rho(X) = \rho(Y) = \rho,$$
$$r(X) = r(Y) = r,$$
$$(-K_X)^3 = (-K_Y)^3 = \deg,$$
$$d(X) = d(Y) = d.$$

Since $\widetilde{X}$ is toric, all its odd cohomologies vanish: $H^3(\widetilde{X}, \mathbb{Q}) = 0$. This implies (see 3.28, 3.29, 3.30):

$$b(Y) = p(X) + \rho(X) - \rho(\widetilde{X}).$$

Put $b = p(X) + \rho(X) - \rho(\widetilde{X})$.

What is left to do is to compute invariants $\rho,\ r,\ \deg, b,\ d$ of X (this is done in section 4), and pick up a unique family of smooth varieties Y with invariants $\rho(Y) = \rho,\ r(Y) = r,\ \deg(Y) = \deg,\ b(Y) = b,\ d(Y) = f$, in the table of [11]. □

The remaining statements in this chapter serve to simplify the computaitons. Picard number of nodal toric Fano threefold is either 1, 2, 3 or 4 (see [18] and table in 5). Hence smooth nontoric Fanos $\rho \geq 5$ (i.e. nontoric variety of degree 28 with $\rho = 5$ and products $\mathbb{P}^1 \times S_{d=11-\rho}$ of the line $\mathbb{P}^1$ with del Pezzo surface S_d of degree $d \leq 5$) has no small toric deformations.

In 55 of 82 cases of singular X the smoothing Y is determined by its invariants $(\rho,\ b,\ r,\ \deg)(Y) = (\rho,\ b,\ r,\ \deg)(X)$. In these cases the routine computation of invariant $d(X)$ may be omitted.

[3]For simplicity of computations, and applications (see 6)

[4]There is an explicit description of nodal toric Fano 3-folds in [18], and more general classifications of all terminal toric Fano 3-folds ([33]) and all Gorenstein toric Fano 3-folds ([32]).

There are eight exceptional sets of invariants (ρ, b, r, deg) corresponding to 17 families of Fanos listed in the following table:

TABLE 1

ρ	deg	$b,\ r$	smooth Y
2	30	0,1	$V_{2.22}[-24], V_{2.24}[-21]$
2	46	0,1	$V_{2.30}[-12], V_{2.31}[-13]$
3	36	0,1	$V_{3.17}[28], V_{3.18}[26]$
3	38	0,1	$V_{3.19}[24], V_{3.20}[28], V_{3.21}[22]$
3	42	0,1	$V_{3.23}[20], V_{3.24}[22]$
4	32	0,1	$V_{4.4}[-40], V_{4.5}[-39]$
2	54	0,2	$V_{2.33}, V_{2.34}$
3	48	0,2	$V_{3.27}, V_{3.28}$

Remark 3.34. Smooth varieties $V_{2.33},\ V_{2.34},\ V_{3.27},\ V_{3.28}$ are toric.

Remark 3.35. In table 1 the number in brackets after smooth Fano Y is its invariant $d(Y)$ (see [12, Proposition 7.35]).

4. The computation

Theorem 4.1 (see e.g. [1]). *Let X be nonsingular and proper (probably not projective) toric variety. Cohomology ring $H^\bullet(X,\mathbb{Q})$ is generated by classes of invariant divisors D_{ρ_i} . The relations in this ring are generated by the so-called Stanley–Reisner relations – for all $J \subset \Sigma^{(1)}$, not contained in any face Δ one has*

$$\prod_{j\in J\subset\Sigma^{(1)}} D_{\rho_j} = 0,$$

and relations implied by the triviality of principal divisors, i.e. for all $m \in M$

$$\sum_i \langle m, \rho_i\rangle D_{\rho_i} = 0.$$

This means that in the cohomology ring of a smooth toric variety all the relations are generated by naive ones: intersection of k different divisors is empty if the corresponding 1-dimensional faces are not contained in one k-dimensional face σ. If they are contained, then the corresponding divisors intersect transversely in $(d-k)$-dimensional orbit corresponding to the face σ.

Lemma 4.2. *Let X_Σ be a smooth toric n-fold. Consider a homogenous system of linear equations*

$$x_{j_1\cdots j_n} = 0,\ \textit{if } \{\rho_{j_1},\ldots,\rho_{j_n}\} \textit{ is not a cone in } \Sigma,$$

$$\sum \langle m, \rho_j\rangle x_{j_1\cdots j_{i-1} j j_{i+1}\cdots j_n} = 0$$

This system has a unique solution up to rescaling. Choose a unique solution that satisfy $x_{j_1\cdots j_n} = 1$, if $\{\rho_{j_1}\ldots\rho_{j_n}\}$ is a cone in Σ. Then the numbers $x_{j_1\cdots j_n}$ are equal to the intersection numbers of divisors $D_{j_1}\cdot\ldots\cdot D_{j_n}$ on X_Σ.

Proposition 4.3. *For Weyl divisor $\sum a_\rho D_\rho$ the condition of local principality in ordinary double point on toric threefold is the following — sum of coefficients at invariant irreducible divisors corresponding to the vertices of the diagonal $\rho_A\rho_C$ of quadrangle $\rho_A\rho_B\rho_C\rho_D$ is equal to the sum at the vertices of $\rho_B\rho_D$:*

$$a_{\rho_A} + a_{\rho_C} = a_{\rho_B} + a_{\rho_D}.$$

Lemma 4.4. *Let X be a nodal toric Fano threefold. Then $\mathrm{Pic}(X)$ is determined from the exact sequence*

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\widetilde{X}) \xrightarrow{\phi} \oplus_{ABCD}\mathbb{Z},$$

where the sum is taken over all basic quadrangles $\rho_A\rho_B\rho_C\rho_D$ for X, $\phi = \oplus_{ABCD}\phi_{ABCD}$, and $\phi_{ABCD}(\sum a_\rho D_\rho) = (a_{\rho_A} - a_{\rho_B} + a_{\rho_C} - a_{\rho_D})$.

Remark 4.5. By virtue of lemmas 4.2 and 4.4, one may effectively compute the intersection theory on $\mathrm{Pic}(X)$ for $\mathbb{Q}$-Gorenstein toric X, admitting a small resolution $f : \widetilde{X} \to X$ (e.g. all nodal threefolds X satisfy this property). Self-intersection D^n of Cartier divisor $D \in \mathrm{Pic}(X)$ is equal to intersection of its pullback $\widetilde{D} = f^*D$ to $\widetilde{X}$. Class group of Weyl divisors is invariant modulo small resolutions, divisor $\widetilde{D}$ is represented by the same Weyl divisor as D (by the pullback).

Therefore to find the intersections on $\mathrm{Pic}(X)$ one need to solve two systems of linear equations: one on intersection numbers $D_{i_1} \cdot \ldots \cdot D_{i_n}$ described in 4.2, and another one — the equations 4.3 cutting $\mathrm{Pic}(X)$ as a subgroup of $\mathrm{Pic}(\widetilde{X})$[5].

Notation. Let M be a integer matrix of size $3 \times v$. Denote by $\Delta(M)$ the convex hull of columns of M. Assume M is chosen in such a way that 0 is contained in the interior of $\Delta(M)$, and none of M's columns lie in the convex hull of the others. By $\mathbb{P}(M)$ denote the toric Fano variety corresponding to the polytope $\Delta(M)$. Let D_i be invariant Weyl divisor corresponding to ith vertice of $\Delta(M)$, and $G_1, \ldots G_\rho$ be the generators of $\mathrm{Pic}(\mathbb{P}(M))$.

In order to compute d, we find first all the intersection numbers of the elements in the base of $\mathrm{Pic}(\mathbb{P}(M))$, and then compute the discriminant. We use 4.5 for the computation of intersection numbers of divisors in $\mathrm{Pic}(\mathbb{P}(M))$ — compute the ring $H^\bullet(\widetilde{\mathbb{P}}(M))$, intersections in Picard group $\mathrm{Pic}(\widetilde{\mathbb{P}}(M))$ of small crepant resolution[6] $\phi : \widetilde{\mathbb{P}}(M) \to \mathbb{P}(M)$, and then intersections in $\mathbb{P}(M)$ is just the restriction from $\widetilde{\mathbb{P}}(M)$.

As an example we produce this computation for case with invariants $(\rho = 2,\ \deg = 30,\ b = 0)$[7].

Case 4.6 $(v = 9,\ f = 10)$.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

[5] The pari/gp script realizing this algorithm is available at http://www.mi.ras.ru/galkin/work/ NodalToric3foldPicard.gp.

[6] We choose arbitrary maximal crepant resolution as explained in 4.5, the answer does not depend on the projectivity of the resolutions.

[7] All the other cases are available at http://www.mi.ras.ru/galkin/work/NodalToric3foldPicard.pdf.

$G_1 = D_1 + D_4 + D_5 + D_8,\ G_2 = -D_1 + D_6 + D_9.$

$$\mathrm{int}(aG_1 + bG_2,\ aG_1 + bG_2,\ aG_1 + bG_2) = (aG_1 + bG_2)^3 = a^3 + 6ba^2 - 2b^3$$
$$-K = G_1 + 2G_2$$
$$d = -24$$

Case 4.7 $(v = 10,\ f = 11)$.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$$

$G_1 = D_7 + D_8 + D_9,\ G_2 = D_2 + D_3 + D_5 - D_6 + D_{10}.$

$$(aG_1 + bG_2)^3 = -2a^3 + 6ba^2 - 3b^3$$
$$-K = 3G_1 + 2G_2$$
$$d = -24$$

Case 4.8 $(v = 9,\ f = 10)$.

$$M = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \end{pmatrix}$$

$G_1 = -D_1 + 2D_3 + D_4 - D_7 + D_8,\ G_2 = D_1 + D_7 + D_9.$

$$(aG_1 + bG_2)^3 = 3ba^2 + 6b^2a$$
$$-K = G_1 + G_2$$
$$d = -21$$

5. The description of toric degenerations of smooth Fano 3-folds

As mentioned in corollary 3.33, for the determination of all the possible types of toric degenerations of Fano threefolds Y, we need to compute the invariants $\rho(X)$, $r(X)$, $\deg(X)$, $b(X)$ (and sometimes $d(X)$) of all nodal toric Fano threefolds. For these computations we used the program[8] based on algorithm described in 4.2, 4.4, 4.5. The results of these computations are exposed in the table 2.

First 4 columns list Fano 3-folds Y and its invariants computed in [12].

In 5th column we list the value of invariant $d(Y)$ for cases when Y is not determined by $(\rho,\ r,\ \deg,\ b)$.

In 6th column we list main combinatorial invariants of toric X (degeneration of Y) — number of vertices, nodes and torus-fixed points.

In 7th column we list the number of toric degenerations X of smooth Y with invariants listed in 6th column.

[8]http://www.mi.ras.ru/galkin/work/NodalToric3foldPicard.gp

Remark 5.1. There is a linear relation 3.30 between $\rho,\ b,\ v,\ p$:

$$v - p = 3 + \rho - b.$$

Remark 5.2. Varieties $V_{2.34},\ V_{3.25},\ V_{3.26},\ V_{3.28},\ V_{4.9}$ are smooth toric varieties that admit degenerations to singular nodal toric varieties. The rest of smooth varieties listed in the table are nontoric.

Remark 5.3. Fano variety $V_{4.13}$ (of degree 26)[9] does not admit small toric degenerations.

Table 2

V_{22}	1	22	0		(13,9,13)	1
V_4	1	32	2		(8,6,6)	1
V_5	1	40	0		(7,3,7)	1
Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
Q	1	54	0		$(5,1,5)$	1
$V_{2.12}$	2	20	3		(14,12,12)	1
$V_{2.17}$	2	24	1		(12,8,12)	1
$V_{2.19}$	2	26	2		(11,8,10)	1
$V_{2.20}$	2	26	0		(11,6,12)	2
$V_{2.21}$	2	28	0		(10,5,11)	2
$V_{2.21}$	2	28	0		(11,6,12)	1
$V_{2.23}$	2	30	1		(9,5,9)	1
$V_{2.22}$	2	30	0		(10,5,11)	1
$V_{2.22}$	2	30	0	$[-24]$	(9,4,10)	1
$V_{2.24}$	2	30	0	$[-21]$	(9,4,10)	1
$V_{2.25}$	2	32	1		(8,4,8)	1
$V_{2.25}$	2	32	1		(9,5,9)	1
$V_{2.26}$	2	34	0		(10,5,11)	1
$V_{2.26}$	2	34	0		(8,3,9)	1
$V_{2.26}$	2	34	0		(9,4,10)	1
$V_{2.27}$	2	38	0		(7,2,8)	1
$V_{2.27}$	2	38	0		(8,3,9)	2
$V_{2.28}$	2	40	1		(7,3,7)	1
$V_{2.29}$	2	40	0		(7,2,8)	1
$V_{2.29}$	2	40	0		(8,3,9)	1
$V_{2.30}$	2	46	0	$[-12]$	(6,1,7)	1
$V_{2.31}$	2	46	0	$[-13]$	(6,1,7)	1
$V_{2.31}$	2	46	0	$[-13]$	(7,2,8)	1
$V_{2.32}$	2	48	0		(6,1,7)	1
$V_{2.34}$	2	54	0		(6,1,7)	1
$V_{3.7}$	3	24	1		(12,7,13)	1

[9] 9Missing in the original version of [11]

Table 2

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
$V_{3.10}$	3	26	0		(11,5,13)	1
$V_{3.11}$	3	28	1		(10,5,11)	1
$V_{3.12}$	3	28	0		(10,4,12)	1
$V_{3.12}$	3	28	0		(11,5,13)	1
$V_{3.13}$	3	30	0		(10,4,12)	2
$V_{3.13}$	3	30	0		(9,3,11)	1
$V_{3.14}$	3	32	1		(8,3,9)	1
$V_{3.15}$	3	32	0		(10,4,12)	1
$V_{3.15}$	3	32	0		(9,3,11)	3
$V_{3.16}$	3	34	0		(8,2,10)	1
$V_{3.16}$	3	34	0		(9,3,11)	1
$V_{3.17}$	3	36	0	[28]	(8,2,10)	2
$V_{3.17}$	3	36	0	[28]	(9,3,11)	1
$V_{3.18}$	3	36	0	[26]	(8,2,10)	1
$V_{3.18}$	3	36	0	[26]	(9,3,11)	1
$V_{3.19}$	3	38	0	[24]	(7,1,9)	1
$V_{3.19}$	3	38	0	[24]	(8,2,10)	1
$V_{3.20}$	3	38	0	[28]	(7,1,9)	1
$V_{3.20}$	3	38	0	[28]	(8,2,10)	1
$V_{3.20}$	3	38	0	[28]	(9,3,11)	1
$V_{3.21}$	3	38	0	[22]	(8,2,10)	1
$V_{3.22}$	3	40	0		(7,1,9)	1
$V_{3.23}$	3	42	0	[20]	(7,1,9)	1
$V_{3.23}$	3	42	0	[20]	(8,2,10)	1
$V_{3.24}$	3	42	0	[22]	(7,1,9)	1
$V_{3.24}$	3	42	0	[22]	(8,2,10)	1
$V_{3.25}$	3	44	0		(7,1,9)	1
$V_{3.26}$	3	46	0		(7,1,9)	1
$V_{3.28}$	3	48	0		(7,1,9)	1
$V_{4.1}$	4	24	1		(12,6,14)	1
$V_{4.2}$	4	28	1		(10,4,12)	1
$V_{4.3}$	4	30	0		(10,3,13)	1
$V_{4.4}$	4	32	0	$[-40]$	(9,2,12)	1
$V_{4.5}$	4	32	0	$[-39]$	(9,2,12)	1
$V_{4.6}$	4	34	0		(10,3,13)	1
$V_{4.6}$	4	34	0		(9,2,12)	1
$V_{4.7}$	4	36	0		(8,1,11)	2
$V_{4.7}$	4	36	0		(9,2,12)	1
$V_{4.8}$	4	38	0		(8,1,11)	1
$V_{4.9}$	4	40	0		(8,1,11)	1

Any smooth Fano threefold not listed in the table does not admit any small toric degenerations, since none of nodal toric Fano threefolds has the proper invariants.

6. Corollaries

We used the classification of smooth Fano 3-folds to compare numerical invariants of X and Y. But one may recover the required parts of this classification used in the proccess of the proof (in case if it would not be known, that is to proof there exists a smooth Fano Y with invariants ρ, deg, d, $b = b(X)$, where X is some known nodal Fano variety (e.g. toric).

We may find some other invariants except of these «classical» ones.

Proposition 6.1. *Let X be a Gorenstein toric Fano variety with isolated singularities. Then there exists a smooth anticanonical section $S \in |-K_X|$, and it is a Calabi–Yau variety.*

Proof. It is a simple corollary of Bertini theorem. □

Proposition 6.2. *Smoothings X_t of Gorenstein Calabi–Yau X_0 are Calabi–Yau varieties.*

Proof. 3.9 implies $h^i(X_t, \mathcal{O}) = 0$ for $0 < i < \dim X_t$. Hence by 3.16 and 3.19 we have the trivializations $K_{\mathcal{X}}|_{X_0} = K_{X_0} = \mathcal{O}_{X_0} \Rightarrow K_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$, so $K_{X_t} = K_{\mathcal{X}}|_{X_t} = \mathcal{O}_{\mathcal{X}}|_{X_t} = \mathcal{O}_{X_t}$. □

Corollary 6.3. *Anticanonical sections X_t are the deformations of anticanonial sections of X_0.*

Proof. If Y_0 is some anticanonical section corresponding to the element $y_0 \in H^0(X_0, -K_{X_0})$, then Y is anticanonical section of X corresponding to $y_0 \otimes 1 \in H^0(X_0, -K_{X_0}) \otimes H^0(\Delta, \mathcal{O}_\Delta) = H^0(\mathcal{X}, -K_{\mathcal{X}})$ (see 3.13), and establishing the required deformation. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}}((-m-1)K_{\mathcal{X}}) \longrightarrow \mathcal{O}(-mK_{\mathcal{X}}) \longrightarrow \mathcal{O}_{\mathcal{Y}}(-mK_{\mathcal{X}}) \longrightarrow 0,$$

vanishings 3.1 and 3.10, 3.11, 3.12, 3.13 (with the similar for $\mathcal{O}(-mK_{\mathcal{X}})$) we deduce that Hilbert polynomial $\mathcal{Y}_t$ does not depend on t, so the family $\mathcal{Y}_t$ is flat. □

Corollary 6.4. *If there exists a smooth anticanonical section of X_0, then general anticanonical section of X_t for general t is smooth.*

Corollary 6.5. *If smooth Fano Y is a smoothing of Gorenstein toric Fano variety with isolated singularities then there exists a smooth anticanonical section $S' \in |-K_Y|$.*

Proof. This is a corollary of 6.1, 6.3 and 6.4. □

For a subvariety $Z \subset X$ (and divisor H) denote by I_H^X the fundamental term of I-series of X (with respect to H), and by $I^{X \to Z}$ — the fundamental term of I-series of Z restricted from X (see [34], [35], [36]). Givental's theorem [34] compute the I-series of smooth complete intersection Z of sections of numerically effective line bundles $\mathcal{O}(Z_i)$, when Z if an almost Fano inside smooth toric X (the similar statement holds for any smooth complete intersection in singular toric variety as well, see [37]). In particular the I-series of toric Fano $X = \mathbb{P}(\Delta)$ of index $r(Y) > 1$ is equal to the series of constant terms Φ_f of Laurent polynomial $f(x) = \sum_{m \in \Delta \cap M} x^m - 1$. Let $[1]g$ denote the coefficient at $1 = x_0$ in Laurent series $f = g(x)$. Then $\Phi_f(t) = [1]e^{tf(x)}$.

Let X be a small toric degeneration of Y and $\phi : \widetilde{X} \to X$ be some small crepant resolution.

Proposition 6.6. *I-series for $I^{Y\to S'}$ restricted from* $\mathrm{Pic}(Y)$ *to* S' *is equal to I-series for* $\widetilde{X}$ *restricted from* $\mathrm{Im}[\mathrm{Pic}(X)\to\mathrm{Pic}(\widetilde{X})]$ *to* $\phi^{-1}(S)\cong S$.

Proof. By 6.1 the general element S of anticanonical linear system $|-K_X|$ of Gorenstein Fano X with isolated terminal singularities is a smooth Calabi–Yau. As we have shown in 6.3, smooth anticanonical sections of X and its smoothing Y lie in the same deformation class. Picard group $\mathrm{Pic}\,X$ is isomorphic to $\mathrm{Pic}\,Y$ by the assumption of smallness. Consider $\mathcal{H}\in\mathrm{Pic}(X)$. Then

$$I^{\widetilde{X}\to S}_{\mathcal{H}_S}=I_{\mathcal{H}^{Y\to S'}_{S'}}.$$

□

Example 6.7. Consider Laurent polynomial

$$f_1=xyz+x+y+z+x^{-1}+y^{-1}+z^{-1},$$

it's Newton polytope $\Delta=\Delta(f)$, and the corresponding toric variety $X=\mathbb{P}(\Delta^\vee)$. One can construct X explicitly: let W be a blowup of a point on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$; then W is almost Fano, but not Fano since the proper transforms of coordinate lines do not intersect $-K_W$; the blowdown X of these lines is a Fano variety with 3 nodes — images of contracted curves, and W is its small crepant resolution. 3 ordinary double points of X correspond to 3 quadrangular faces (xyz,x,y,z^{-1}), (xyz,x,z,y^{-1}) and (xyz,y,z,x^{-1}). Anticanonical degree of X is the same as W's i.e. $\deg(X)=2^3\cdot 6-8=40$. Let Y be a Fano smoothing of X. Consider general Laurent polynomial with Newton polytope Δ:

$$f_a=\sum a_mx^m=a_{xyz}xyz+a_xx+a_yy+a_zz+a_{x^{-1}}x^{-1}+a_{y^{-1}}y^{-1}+a_{z^{-1}}z^{-1}.$$

It corresponds to the divisor $\sum b_mD_m\in\mathrm{Pic}(W)\otimes\mathbb{C}$, such that $a_m=\exp 2\pi i b_m$. This divisor is a pullback of Cartier divisor on X, if its coefficients satisfy 3 conditions of local principality

$$\begin{aligned}b_{xyz}+b_{x^{-1}}&=b_y+b_z,\\ b_{xyz}+b_{y^{-1}}&=b_x+b_y,\\ b_{xyz}+b_{z^{-1}}&=b_x+b_z.\end{aligned}$$

Principal divisors are

$$(b_x+b_y+b_z)xyz+b_xx+b_yy+b_zz-b_xx^{-1}-b_yy^{-1}-b_zz^{-1}.$$

X has index 3, and its Picard group is generated by $-D_{xyz}+D_{x^{-1}}+D_{y^{-1}}+D_{z^{-1}}$. Modulo principal divisors Laurent polynomial corresponding to α-multiple of a generator of $\mathrm{Pic}(X)$ is equal to $f_t=t(xyz+x+y+z+x^{-1}+y^{-1}+z^{-1})$, $t=\exp\pi i\alpha$.

By the virtue of [34] $I^W_{-K_W,1}(t)=\Phi_{f+\alpha}(t)$, i.e. I-series of W is equal to Φ_{f_1} up to renormalization[10]. Let's compute Φ_{f_1}. The products of monomials $\prod_{n_i}(x^{m_i})^{n_i}$ gives a nonzero summand to the series of constant terms if $\sum n_im_i=0$; in our case put $n_{xyz}=d,\ n_x=a,\ n_y=b,\ n_z=c$. Then $n_{x^{-1}}=a+d$, $n_{y^{-1}}=b+d$, $n_{z^{-1}}=c+d$. Hence

$$\Phi_{f_1}=\sum_{a,b,c,d\geq 0}\frac{(2a+2b+2c+4d)!}{a!b!c!d!(a+d)!(b+d)!(c+d)!}t^{2a+2b+2c+4d}=1+6t^2+114t^4+2940t^6+87570t^8+\ldots \tag{6.8}$$

[10] As we will show below index of W and X is equal to 2, hence renormalization is trivial: $\alpha=0$, $\widetilde{t}=t$.

By 6.5 general anticanonical section $S' \in |-K_Y|$ is smooth. Applying the proposition 6.6, we conclude that the restricted from Y regularized I-series $I^{Y \to S'}_{-K_Y,1}$ for smooth anticanonical section of Y is equal to Φ_f.

Therefore we computed the I-series of smoothing Y of X not using the geometry of Y. It is easy to check that Y is a Fano variety V_5, because it is unique Fano 3-fold with invariants $(\rho,\ r,\ \deg,\ b) = (1, 2, 40, 0)$. Since V_5 is a section of Grassmanian $G(2,5)$ by three hyperplanes, its I-series may be computed by applying the quantum Lefschetz formula [35] to the I-series of $G(2,5)$ provided in [6], [38]:

$$I_{G(2,5)} = \sum_{d\geq 0} \frac{t^d}{(d!)^2} \sum_{d\geq j_2\geq j_1\geq j_0=0} \frac{1}{(d-j_2)!\prod_{i=2}^{3}((d-j_{i-1})!(j_{i-1}-j_{i-2})!j_{i-1}!)}.$$

Applying quantum Lefschetz to the I-series of Grassmanian $I_{G(2,5)}$ one indeed deduces is 6.8.

7. Generalizations

Unfortunately only half of nontoric Fano threefolds are smoothings of nodal toric. In particular the only one Fano threefold of principal series admit a small toric degeneration (it is the variety V_{22}). Note that for nodal toric varieties it is easy to prove projective normality and the smoothness of general anticanonical section, and one can show these properties holds for the smoothing as well (as in 6.3). All the smoothings Y we obtained are rational. The same method could be applied to obtain more general class of smoothings, if we consider not only the tproc varieties, but also complete intersections inside them (with Gorenstein terminal singularities) — these varieties also admit a smoothing (3.23), and there are similar relations between invariants of the smoothing and the degeneration, and it is not so hard to compute the cohomologies of such varieties ([39]), Hilbert polynomial, and Gromov–Witten theory.

But birational class of complete intersection in toric variety is arbitrary. Many of nondegenerating to nodal toric threefolds are themselves the complete intersections in weighted projective spaces. Batyrev and Kreuzer found all nodal half-anticanonical hypersurfaces in toric fourfolds of index 2: there are around 160 of them, and 100 are cones over the toric varieties studied in this paper, the remaining 60 cases cover almost alll nondegenerating to toric Fano varieties.

Another direction for generalizations is toric varieties with arbitrary Gorenstein singularities. For a pair of nonterminal Gorenstein toric Fano threefolds $\mathbb{P}(\Delta_{16})$, $\mathbb{P}(\Delta_{18})$ Przhyalkovskii constructed ([36]) a pair of Laurent polynomials f_{16}, f_{18} with Newton polytopes coinciding with the corresponding fan polytopes Δ_{16}, Δ_{18} such that these polynomials are weak Landau–Ginzburg models mirror symmetric to Fano varieties of principal series V_{16} and V_{18}; so it is possible that toric degenerations mothod works for a larger class of singularities (all Gorenstein?), altough we don't know if the pairs $\mathbb{P}(\Delta_{16}, V_{16})$ and $\mathbb{P}(\Delta_{18}, V_{18})$ are the degenerations.

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