Bootstrap Program for CFT in D>=3

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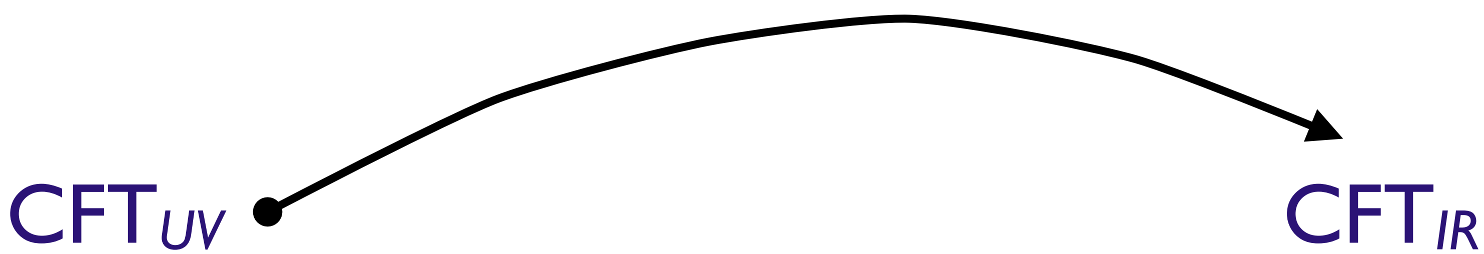
ENS Paris & CERN

Physical Origins of CFT

RG Flows:



Fixed points = CFT

[Rough argument: $T^{\mu}_{\mu} = \beta(g)\mathcal{O} \to 0$ when $\beta(g) \to 0$]

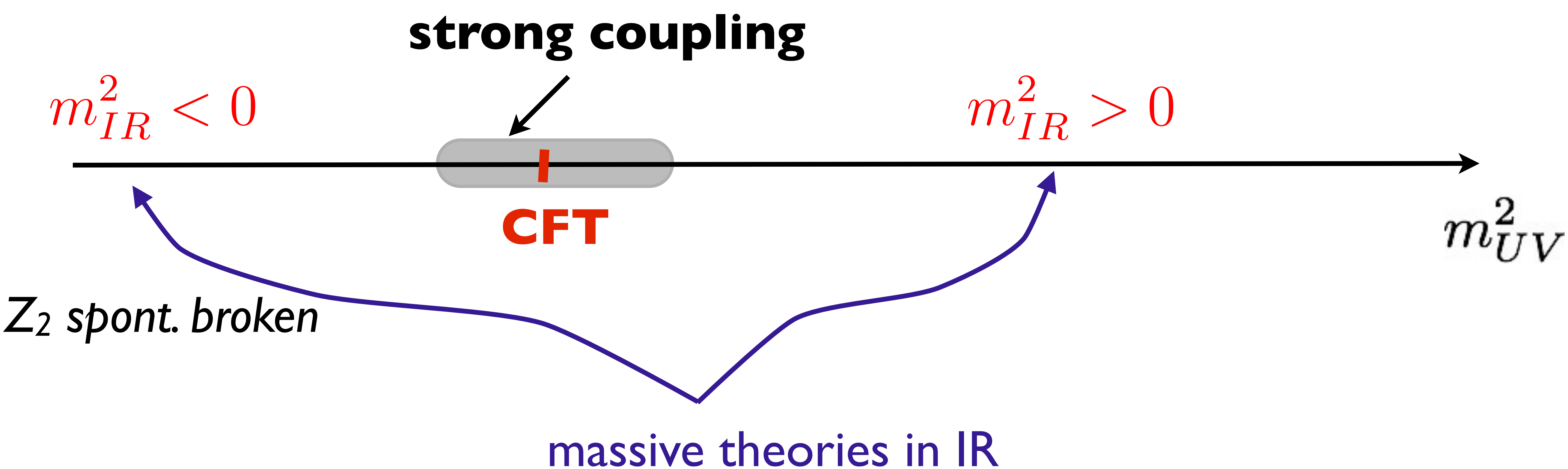
3D Example

CFT_{UV} = free scalar $(\partial\phi)^2$

Z_2-preserving perturbation: $m^2\phi^2 + \lambda\phi^4 \; [+\kappa\phi^6]$ $\quad m, \lambda \ll \Lambda_{UV}$

$$m_{IR}^2 = m_{UV}^2 + O\left(\frac{\lambda^2}{16\pi^2}\right)$$

Phase diagram:

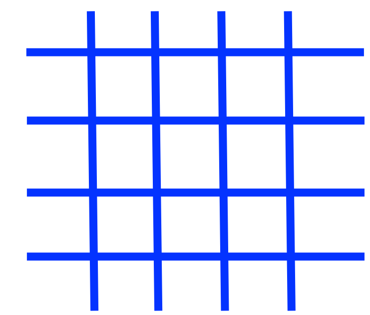


Universality

- Any same-symmetry Lagrangian (e.g. k≠0) can flow to the same CFT_{IR}

- Can even start from a lattice model e.g. 3D Ising model:

$$Z = \exp\left[-\frac{1}{T}\sum_{\langle ij\rangle}\sigma_i\cdot\sigma_j\right]$$



Near T_c the spin-spin correlation length $\xi(T) \to \infty$
$\Rightarrow$ lattice artifacts go away

Continuum limit @ $T=T_c$ is the same CFT_{IR} as on the previous slide

Beyond Lagrangians

Strongly coupled CFTs can usually be realized as endpoints of RG flows from weakly coupled, Lagrangian theories

Exception: N=(2,0) 6D theory of multiple M5 branes

By itself, a CFT **generically** cannot be described by a Lagrangian

Strongly coupled Lagrangian ≈ No Lagrangian

Exceptions:

a) Weakly coupled CFTs, like $\lambda\varphi^4$ in D=4-ε (WF fixed point) $\mathrm{CFT}_{UV} \longrightarrow \mathrm{CFT}_{IR}$

b) Theories a la N=4 SYM $\mathcal{L} = \frac{1}{g^2}[\ldots]$ $\beta(g) \equiv 0 \; \forall g$

One parameter family of CFTs:

free $\bullet\!\longrightarrow g$

weakly coupled

strongly coupled, defined by analytic continuation

Beyond AdS

For many people, CFT in D>=3 has become inseparable from AdS/CFT

Does any CFT has an AdS dual (string σ -model with AdS factor in the target space)?

Is duality practical away from the large N limit?

Effective holography:

Put any field content in the AdS bulk, compute correlators on the boundary

Theory in the bulk is only effective (e.g. includes gravity)
$\Rightarrow$

defines only an `effective CFT', to first order in 1/N expansion

$$N \sim R_{AdS}/L_{Pl}$$

CFT - intrinsic definition

I. *Basis of local operators O_i with scaling dimensions Δ_i*

[including stress tensor $T_{\mu\nu}$ of Δ_T=4; conserved currents J_μ of Δ_J=3]

$$O_\Delta \xrightarrow{P} O_{\Delta+1} \xrightarrow{P} O_{\Delta+2} \xrightarrow{P} \ldots$$

derivative operators **(descendants)**

K_μ = special conformal transformation generator, [K]=-1

$$K_\mu \leftrightarrow 2x_\mu(x\cdot\partial) - x^2\partial_\mu \qquad \text{cf. } P_\mu \leftrightarrow \partial_\mu$$

$$O_\Delta \xleftarrow{K} O_{\Delta+1} \xleftarrow{K} O_{\Delta+2} \xleftarrow{K} \ldots$$

In unitary theories dimensions have lower bounds:

$$\Delta \geq \ell + D - 2 \ (\geq D/2 - 1 \text{ for } \ell = 0)$$

So each multiplet must contain the lowest-dimension operator:

(primary)

$$K\mu \cdot O_\Delta(0) = 0$$

At x≠ 0: $[K_\mu, \phi(x)] = (-i2x_\mu\Delta - 2x^\lambda\Sigma_{\lambda\mu} - i2x_\mu x^\rho\partial_\rho + ix^2\partial_\mu)\phi(x)$

Ward identities for correlation functions:

$$X \cdot \langle \ldots \rangle = 0 \qquad X = (D, P_\mu, M_{\mu\nu}, K_\mu)$$

For 2- and 3-point functions suffice to solve the x-dependence:

normalization

$$\langle O_i(x) O_j(0) \rangle = \frac{\delta_{ij}}{(x^2)^{\Delta_i}}$$

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k} |x_{13}|^{\Delta_i+\Delta_k-\Delta_j} |x_{23}|^{\Delta_j+\Delta_k-\Delta_i}}$$

2. "coupling constants"

= OPE coefficients

= structure constants of the operator algebra

Operator Product Expansion

$$O_i(x)O_j(0) = \lambda_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\{O_k(0)+\ldots\}$$

can be determined by plugging OPE into 3-point function and matching on the exact expression

$$\frac{1}{2}x^\mu\partial_\mu O_k + \alpha x^\mu x^\nu \partial_\mu\partial_\nu O_k + \beta x^2\partial^2 O_k + \ldots$$

Four point function

Ward identity constrains it to have the form:

$$\langle\phi\phi\phi\phi\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}$$

$$\boxed{u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}}$$

Using OPE can say more:

$$\left\langle \begin{matrix} \phi(x_1) & \phi(x_3) \\ \phi(x_2) & \phi(x_4) \end{matrix} \right\rangle = \sum \lambda_{\phi\phi i}^2 |x_{12}|^{\Delta_i - 2\Delta_\phi} |x_{34}|^{\Delta_i - 2\Delta_\phi} \langle \{O_i(x_2) + \cdots\}\{O_i(x_4) + \cdots\}\rangle$$

$$= \sum \lambda_{\phi\phi i}^2 \frac{G_{\Delta_i,\ell_i}(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}$$

conformal blocks

$$\boxed{g(u,v) = \sum \lambda_{\phi\phi i}^2 G_{\Delta_i,\ell_i}(u,v)}$$

Crossing symmetry

$$\langle\phi\phi\phi\phi\rangle = \frac{g_s(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} = \frac{g_t(v,u)}{|x_{14}|^{2\Delta}|x_{23}|^{2\Delta}}$$

$$g_s(u,v) = \sum \lambda^2_{\phi\phi i} G_{\Delta_i,\ell_i}(u,v) \qquad g_t(v,u) = \sum \lambda^2_{\phi\phi i} G_{\Delta_i,\ell_i}(v,u)$$

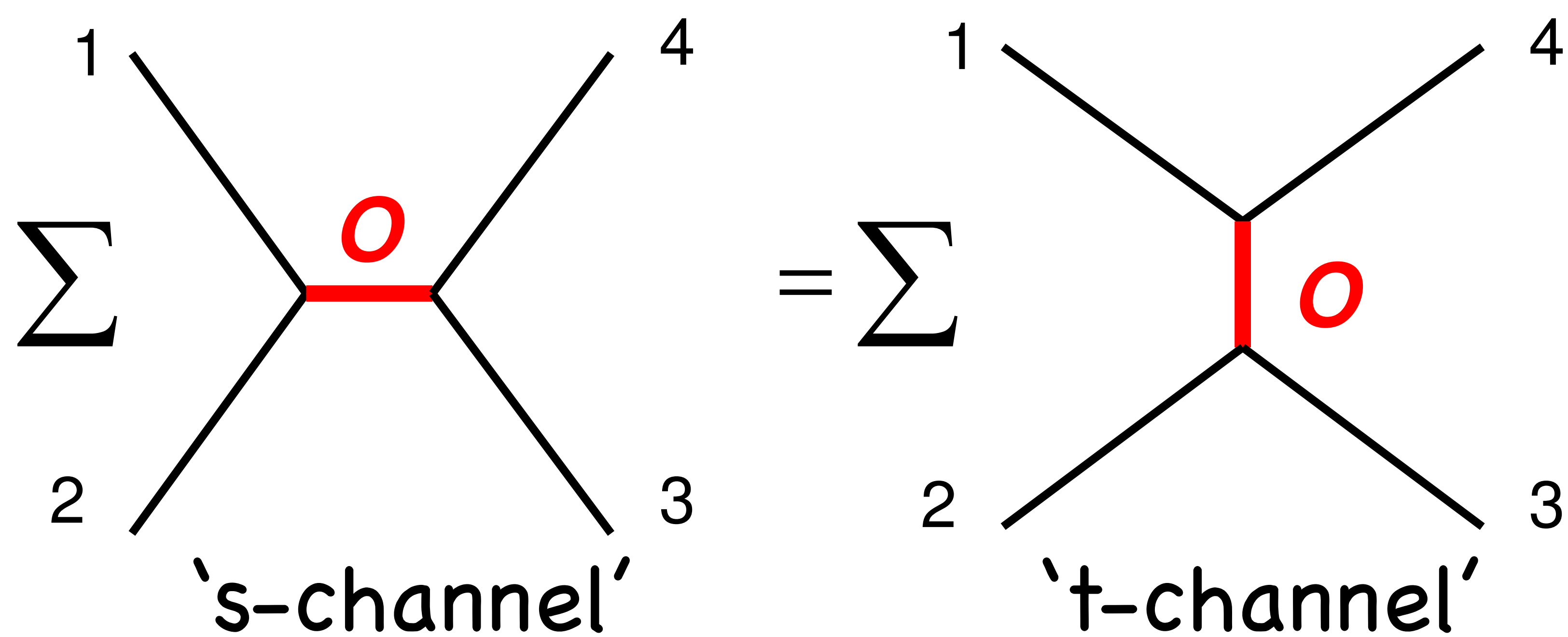
But: $g_t(v,u) = (v/u)^{\Delta_\phi} g_s(u,v)$

This is a consistency condition for the CFT data

[Nontrivial because not satisfied term by term]

Conformal bootstrap

Ferrara,Gatto,Grillo 1973
Polyakov 1974



$$\sum_i \lambda_{12i}\lambda_{34i}[\ldots] = \sum_i \lambda_{14i}\lambda_{23i}[\ldots]$$

Do solutions of this equation, imposed on all four point functions, provide a classification of CFTs?

A bit like classifying Lie algebras...

D=2 success story

- In D=2 $(P_\mu, K_\mu, M_{\mu\nu}, D) \rightarrow$ Virasoro algebra

$\Rightarrow$ New lowering operators L_{-n}, n=2,3,...

Virasoro multiplet = $\bigoplus_{n=1}^{\infty}$ (Conformal multiplets)

- Central charge c<1 + unitarity $\Rightarrow$

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, \ldots$$

[Friedan,Qiu, Shenker]

- Primary dimensions in these "minimal models" are also fixed:

$$\Delta_{r,s} = \frac{(r + m(r-s))^2 - 1}{2m(m+1)} \qquad 1 \leq s \leq r \leq m-1$$

-Finally, knowing dimensions, OPE coefficients can be determined by **bootstrap**

[Belavin, Polyakov, Zamolodchikov], ...

D>=3 always looked a bit hopeless...

$$\sum_i \lambda_{12i}\lambda_{34i}G(\Delta_i,\Delta_{ext}|u,v) = \sum_i \lambda_{14i}\lambda_{23i}G(\Delta_j,\Delta_{ext}|v,u)$$

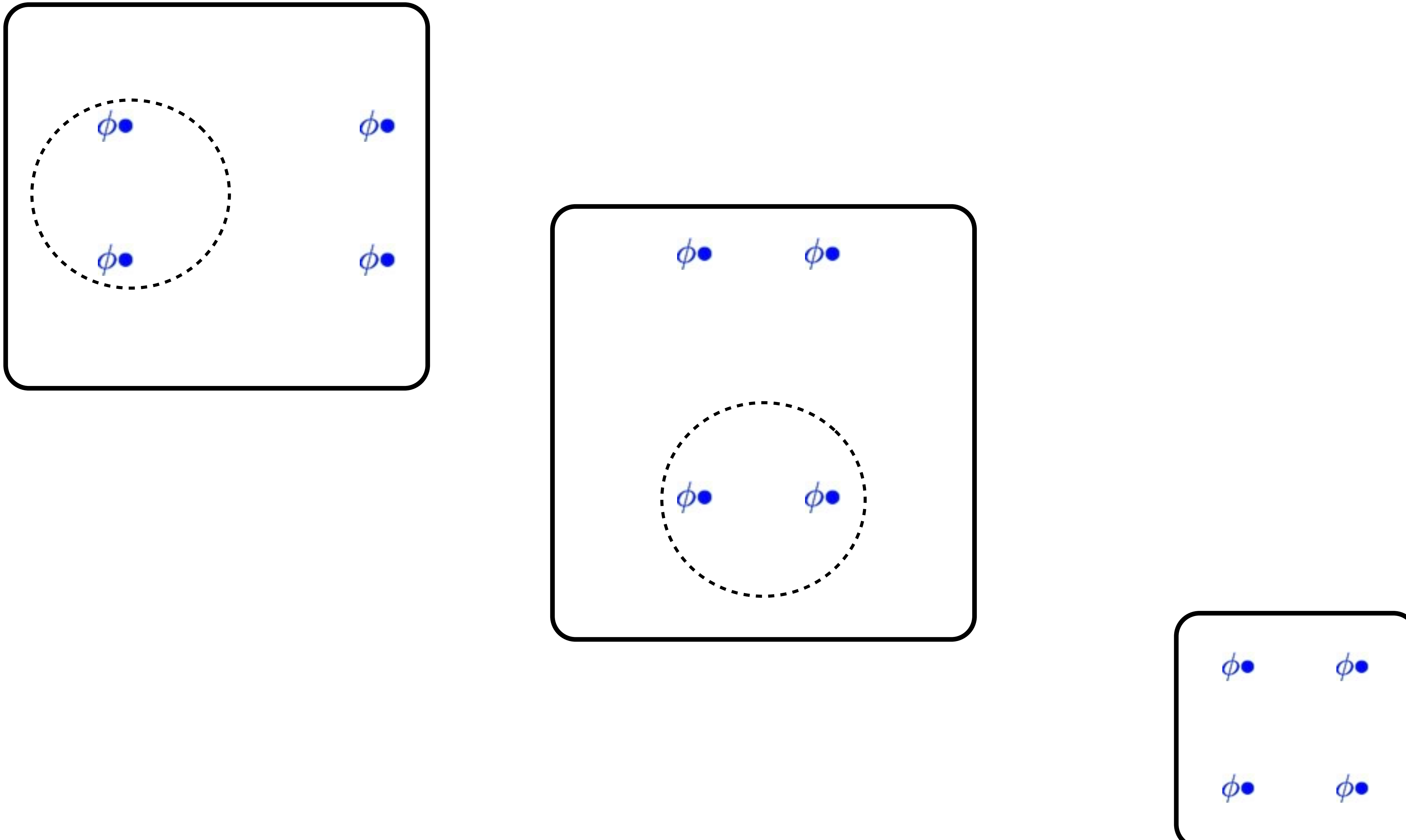
- Infinite system for infinite # of unknowns

- # of primaries grows exponentially with dimension:

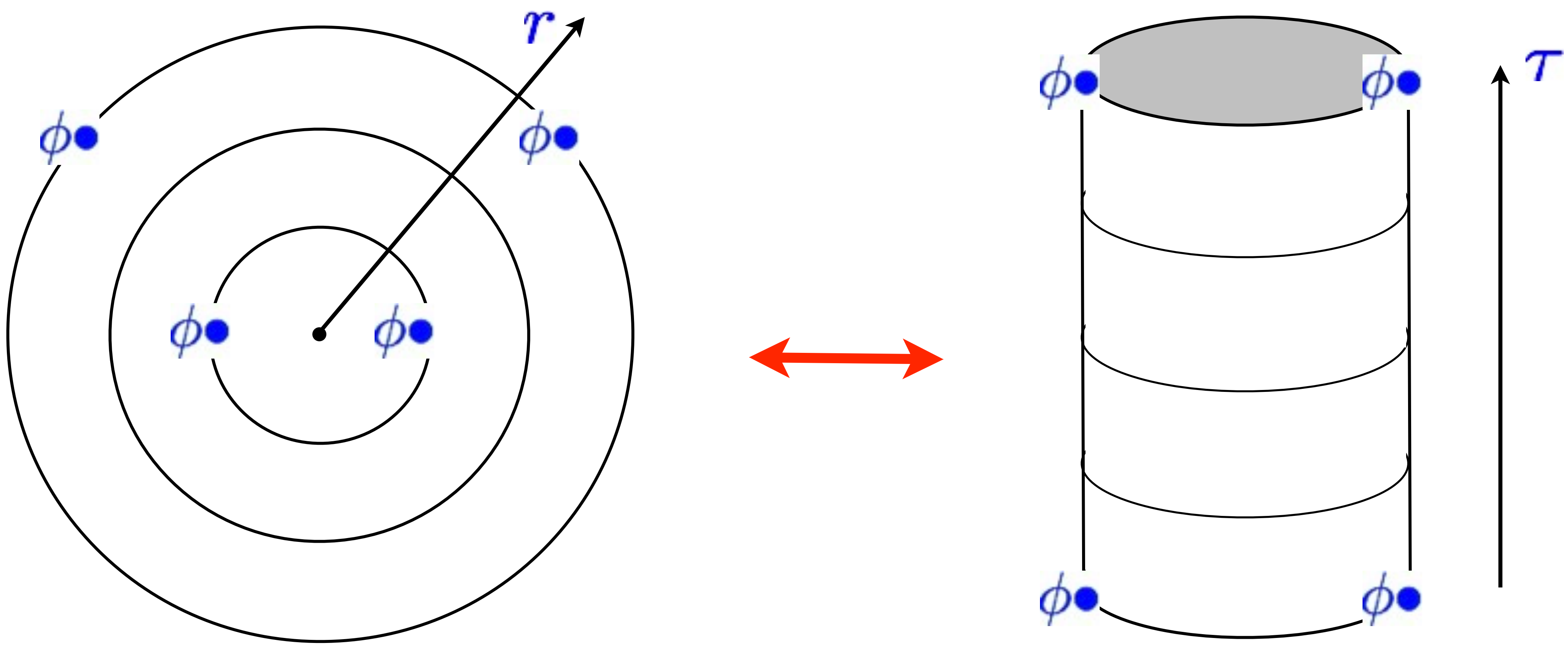
$$\#(\Delta < E) \sim \exp(Const.E^{1-1/D})$$

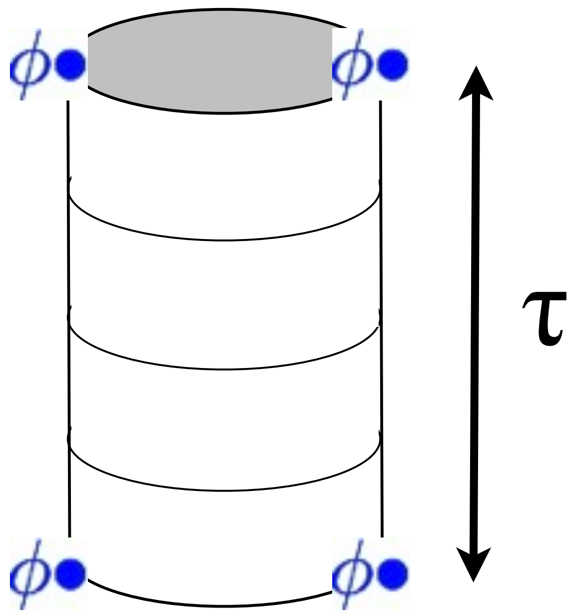
Expansion parameter? Convergence?

Convergence of OPE decomposition



Mapping to the cylinder (Radial quantization)





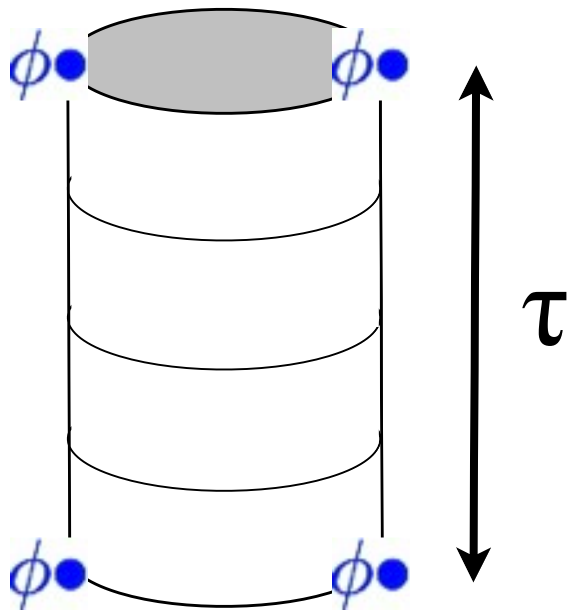
$$\langle 0|\phi\phi\phi\phi|0\rangle = \sum_{E_n} \langle 0|\phi\phi|n\rangle e^{-E_n \tau} \langle n|\phi\phi|0\rangle$$

States on the cylinder are in one-to-one correspondence with CFT local operators (State-operator correspondence)

$$|\Delta\rangle \leftrightarrow O_\Delta \qquad\qquad E_n = \Delta + n, \quad n = 0, 1, 2, \ldots$$

$$\langle 0|\phi\phi\phi\phi|0\rangle = \sum_{\Delta} |\langle 0|\phi\phi|\Delta\rangle|^2 e^{-\Delta\tau} \underbrace{(1 + \sum_{n=1}^{\infty} c_n e^{-n\tau})}_{\text{conformal block}}$$

OPE coefficient



$$\langle 0|\phi\phi\phi\phi|0\rangle = \sum_{\Delta} |\langle 0|\phi\phi|\Delta\rangle|^2 e^{-\Delta\tau}(1+\sum_{n=1}^{\infty} c_n e^{-n\tau})$$

In the limit $\tau \to 0$: $$\langle 0|\phi\phi\phi\phi|0\rangle \sim \frac{1}{\tau^{2\Delta_\phi}} \times \frac{1}{\tau^{2\Delta_\phi}}$$

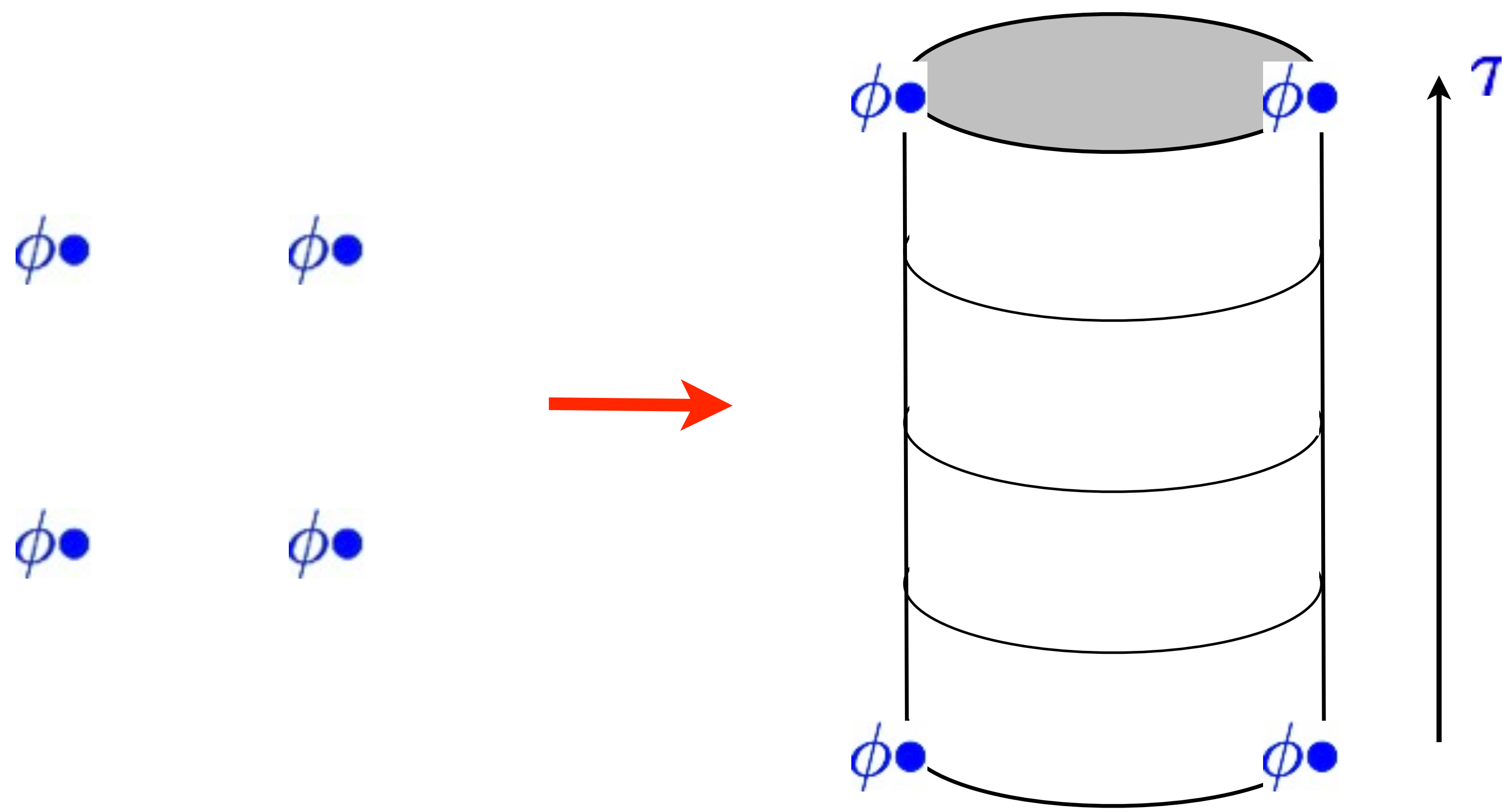
⇒ OPE coefficient asymptotics: $$|\langle 0|\phi\phi|\Delta\rangle|^2 \sim \frac{\Delta^{4\Delta_\phi - 1}}{\Gamma(4\Delta_\phi)}$$

⇒ At any finite $\tau>0$ the series converges exponentially fast:

$$\langle 0|\phi\phi\phi\phi|0\rangle|_{\Delta \geq \Delta_*} \lesssim \frac{\Delta_*^{4\Delta_\phi}}{\Gamma(4\Delta_\phi + 1)} e^{-\Delta_* \tau}$$

[Pappadopulo, S.R., Espin, Rattazzi]

[Pappadopulo, S.R., Espin, Rattazzi]



$$e^{-\tau} = 3 - 2\sqrt{2} \approx 0.17$$

small parameter!

Still the full bootstrap system looks difficult...

Focus on the 4-point function of the lowest dimension scalar:

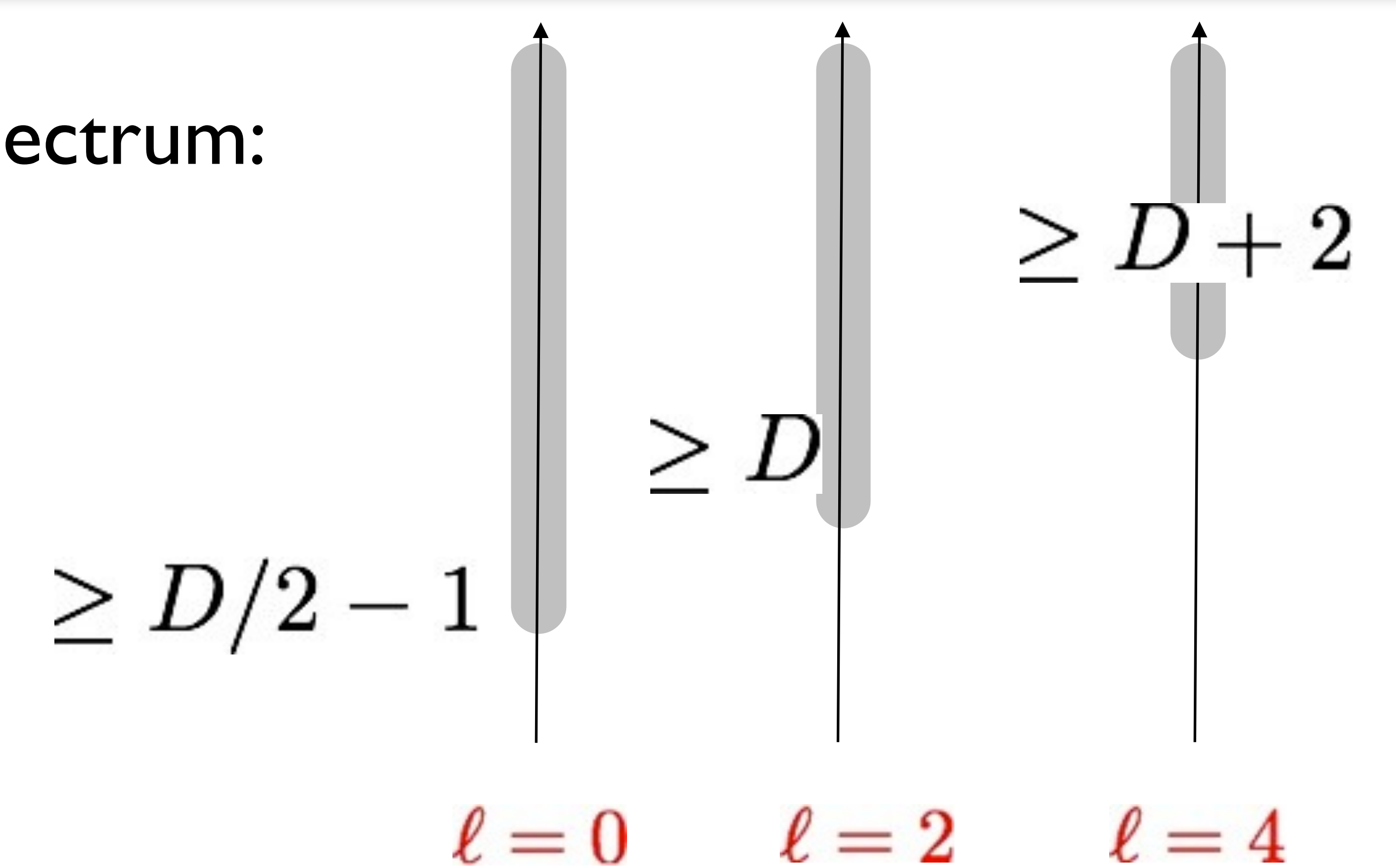
$$\phi \times \phi = 1 + {}''\phi^{2}{}'' + \ldots \qquad \text{spin 0}$$

$$+T_{\mu\nu} + \ldots \qquad \text{spin 2}$$

+ spins 4,6,...

lowest dimension scalar in this OPE

Allowed spectrum:



Bootstrap equation:

$$v^{\Delta_\phi} + \sum_{l=0,2,\dots} \sum_{i=1}^{\infty} X_{\ell,i}\; v^{\Delta_\phi} G_{\ell,\Delta_i}(u,v) = (u \leftrightarrow v)$$

unknowns

$$X_{\ell,i} \geq 0$$ (square of a real OPE coefficient)

E.g. free scalar field is a solution:

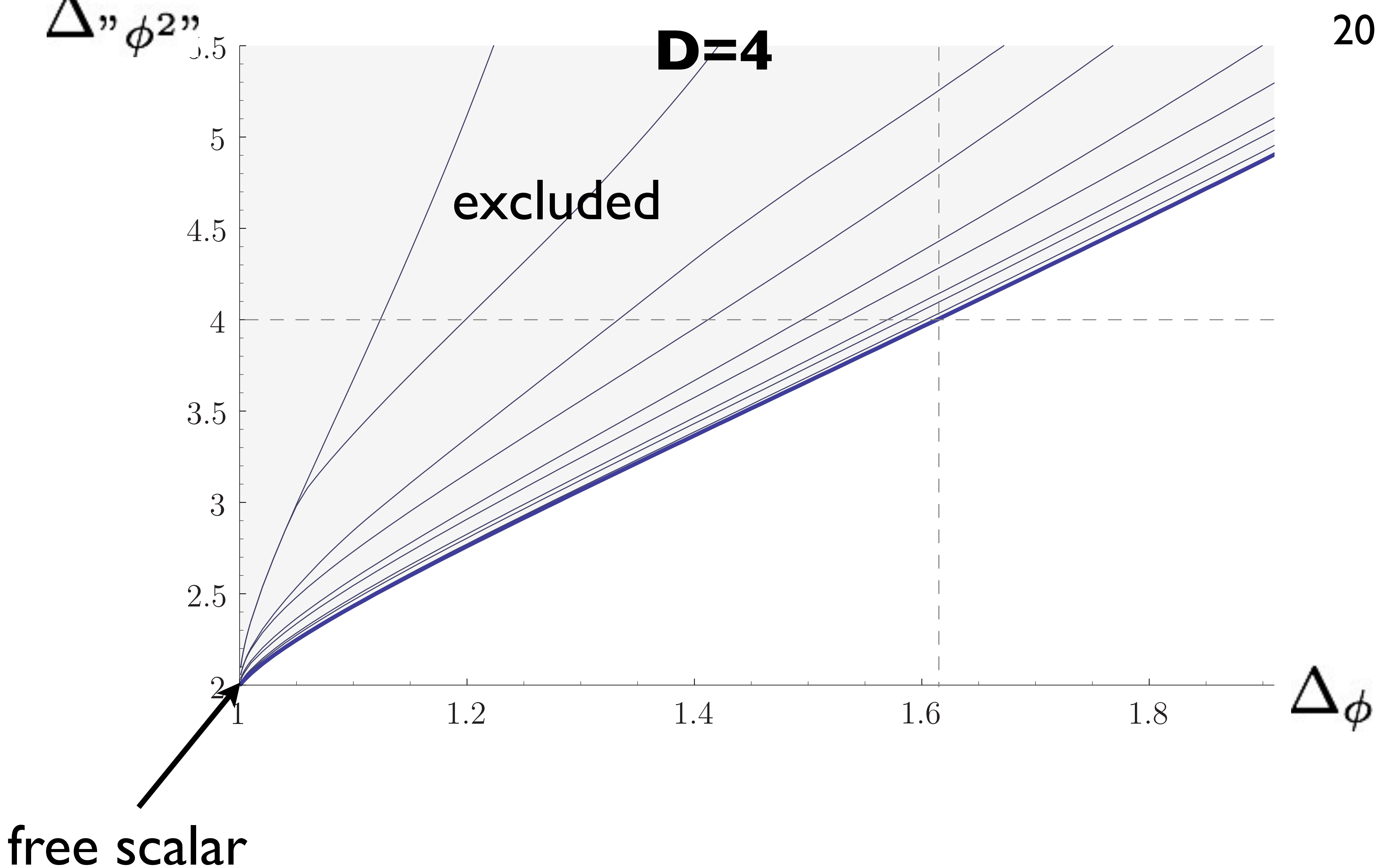
$$\Delta_\phi = 1 \quad (D=4)$$

$$\Delta_l = l + D - 2$$ (one field per spin in the OPE)

$$X_l = \frac{(l!)^2}{(2l)!}$$

Upper bound on the dimension of "φ^2"

Rattazzi, S.R., Tonni, Vichi 2008
S.R., Vichi 2009
…, Poland, Simmons-Duffin, Vichi 2011



$$v^{\Delta_\phi} + \sum_{l=0,2,\dots} \sum_{i=1}^{\infty} X_{\ell,i}\, v^{\Delta_\phi} G_{\ell,\Delta_i}(u,v) = (u \leftrightarrow v)$$

Expand the bootstrap equation around the square configuration up to a fixed order:

$$\sum_{l=0,2,\dots} \sum_{i=1}^{\infty} X_{\ell,i} \vec{V}_{\ell,\Delta_i} = \vec{V}_0 \qquad X_{\ell,i} \geq 0$$

O(100) components

Δ_i: put an upper cutoff and discretize - get a finite system

No solutions without low-dimension scalars in the spectrum Rattazzi, S.R., Tonni, Vichi 2008

Some methods avoid discretization and upper cutoff on Δ (only on spin) Poland, Simmons-Duffin, Vichi 2011

Direction 1. “Carving out the space of CFTs”

- Bounds on the OPE scalar spectrum in presence of global symmetry of supersymmetry

Poland, Simmons-Duffin 2010,
Rattazzi, S.R., Vichi 2010
Vichi 2011
Poland, Simmons-Duffin, Vichi 2011

- Bounds on the OPE coefficients and central charges (as functions of operator dimensions)

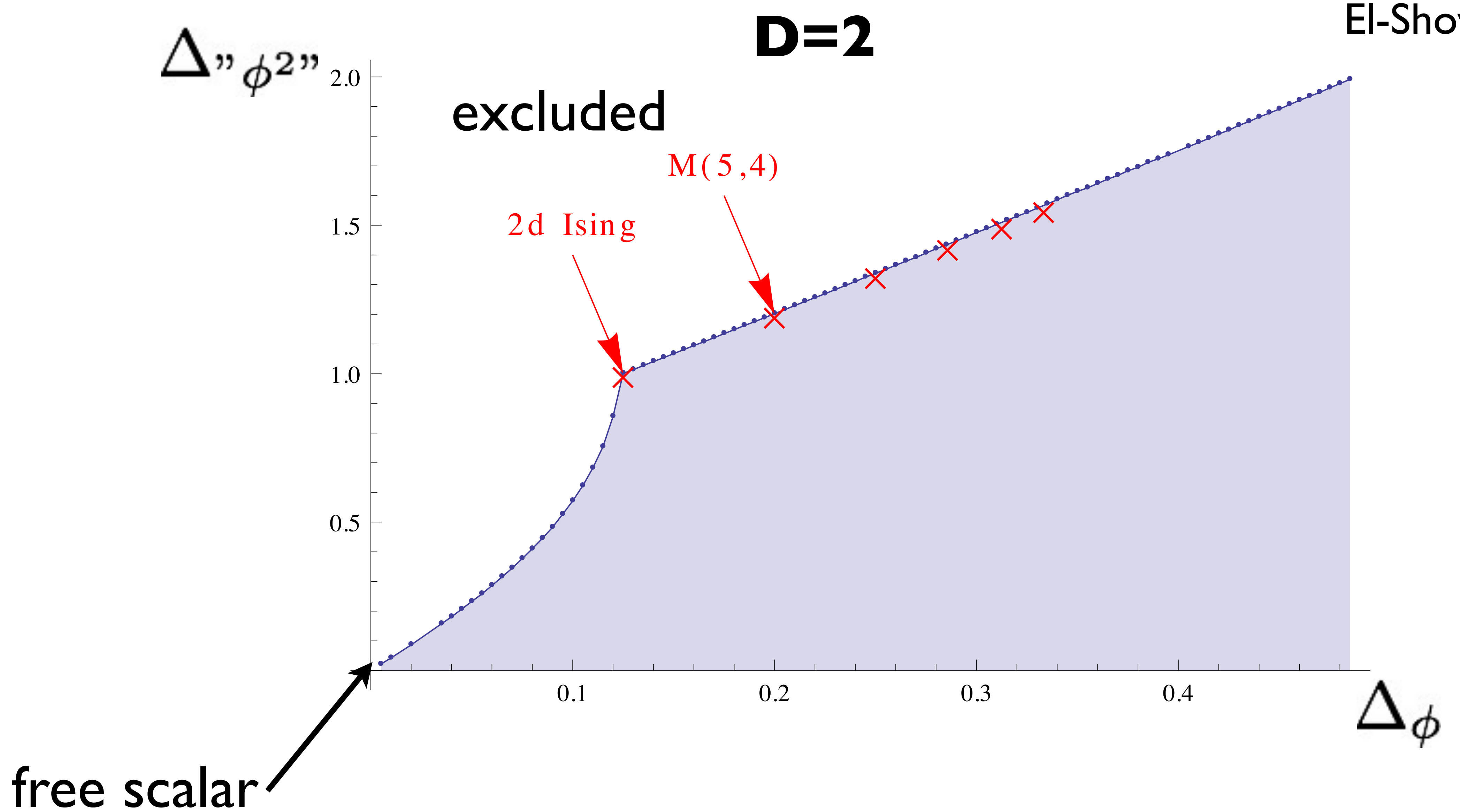
Caracciolo, S.R 2009,
Poland, Simmons-Duffin 2010,
Rattazzi, S.R., Vichi 2010

- Bounds on the CFT data in presence of a boundary

Liendo, Rastelli, van Rees 2012

Direction 2. “Looking for kinks”

S.R., Vichi 2009
El-Showk, Paulos 2012



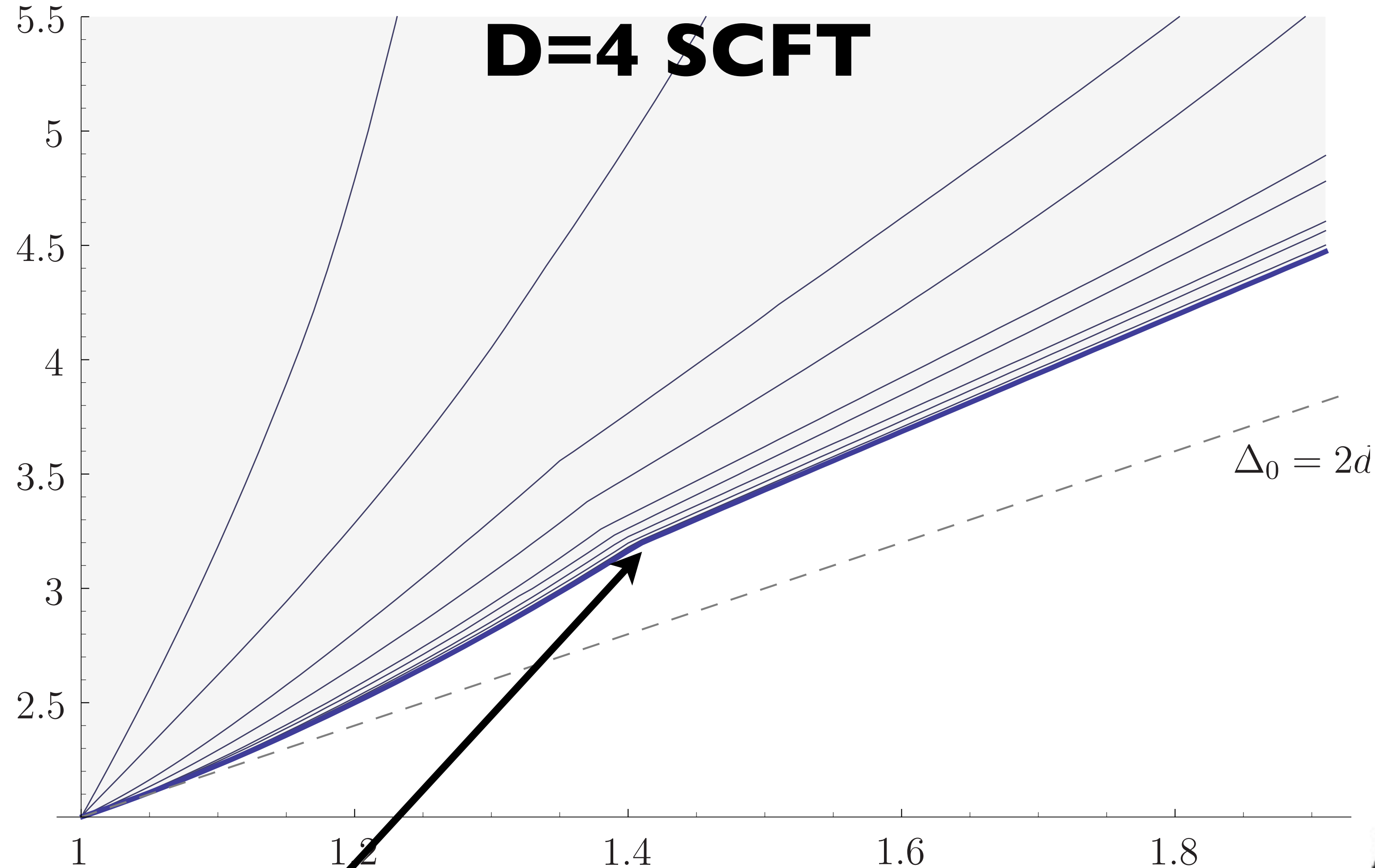
It could be that some special theories saturate bounds and/or live at corner points

SUSY kink

Poland, Simmons-Duffin, Vichi 2011

$\Delta_{\Phi^\dagger\Phi}$

first scalar superprimary in $\Phi^\dagger \times \Phi$ OPE



Δ_Φ

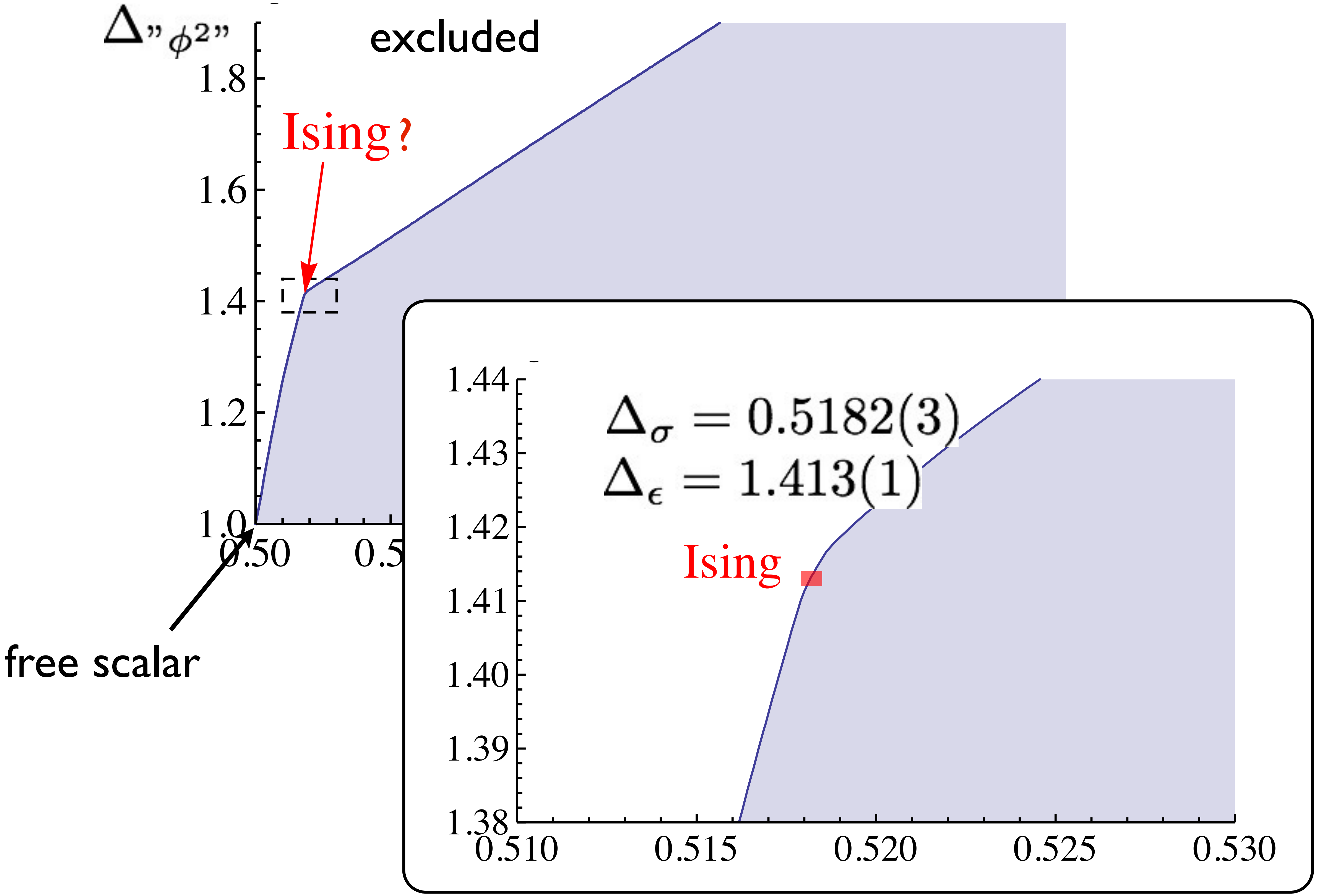
What is this theory?

[Conjecturally, Φ^2=0 in its chiral ring]

chiral primary; uncharged under global symmetry

In D=3 the kink is still there:

[El-Showk, Paulos, Poland, Simmons-Duffin, S.R,,Vichi'2012]



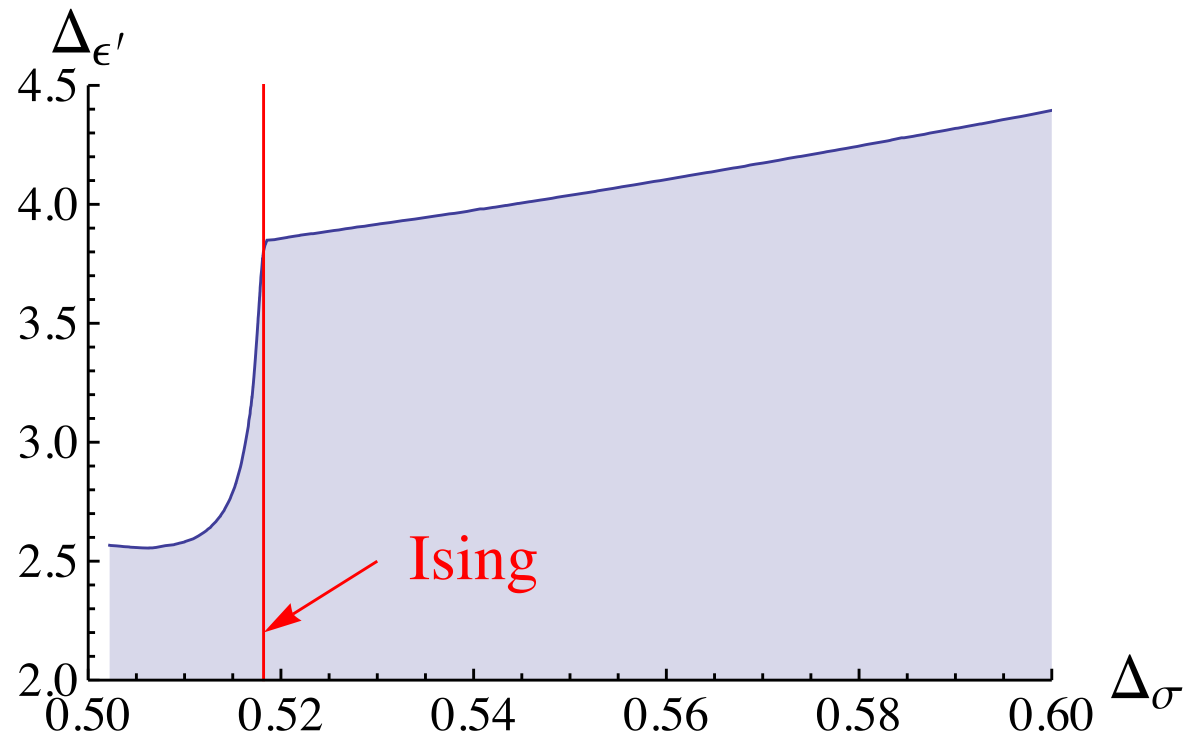
Interesting things happen near 3D Ising kink:

[El-Showk, Paulos, Poland, Simmons-Duffin, S.R,, Vichi'2012]

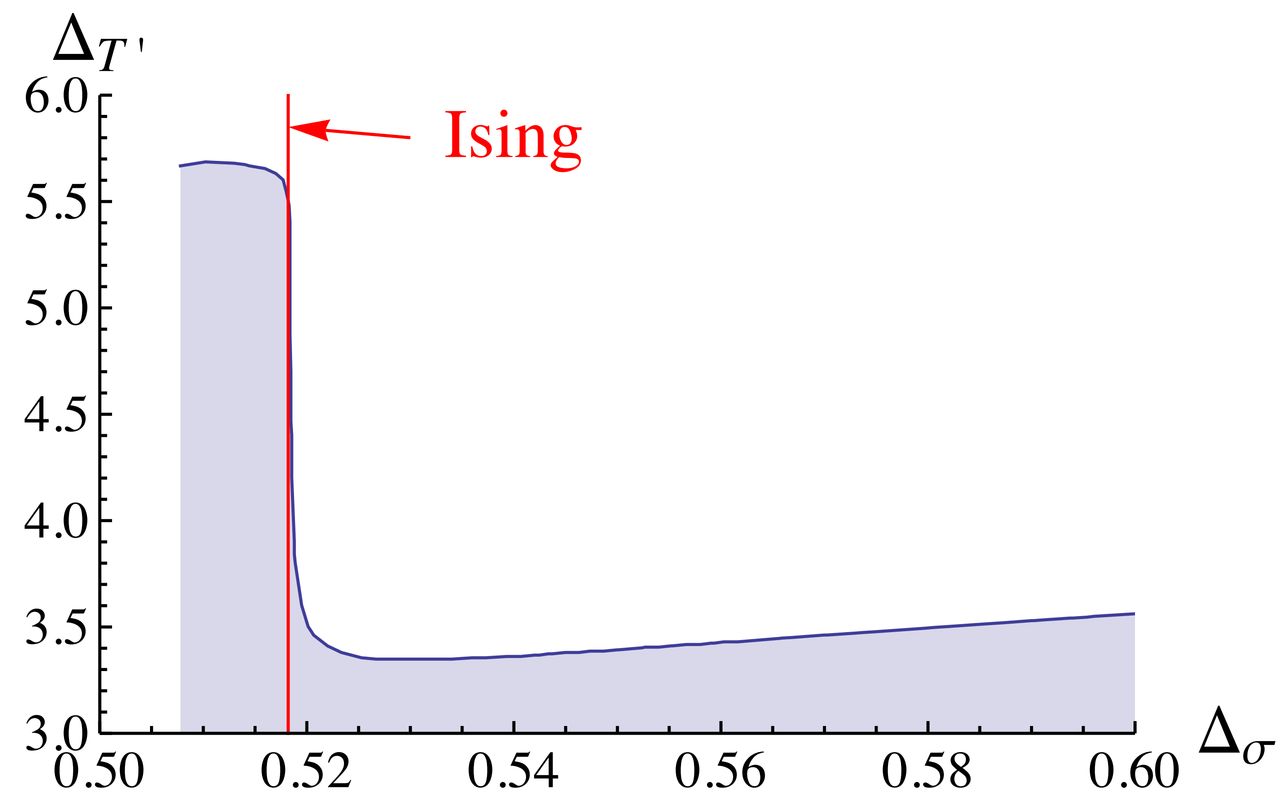
$$\sigma \times \sigma = 1 + \epsilon + \epsilon' + \ldots$$

$$\Delta_{\epsilon'} = 3.84(4)$$

fix to maximally allowed



$$+T_{\mu\nu} + T'_{\mu\nu} + \ldots$$



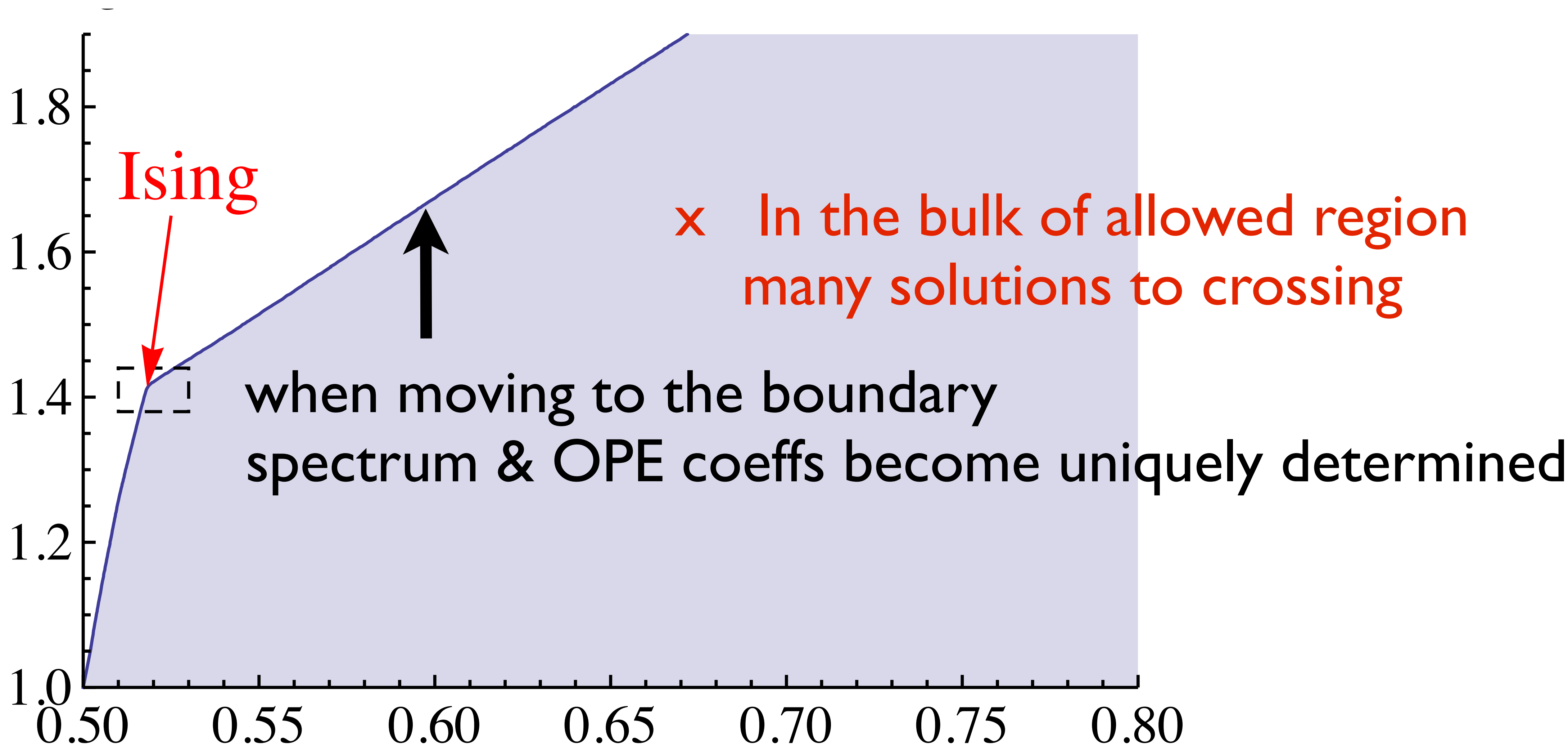
Future Directions & Open problems

1. Extend the crossing symmetry analysis to different external states
   - stress tensor and currents
   - fermions

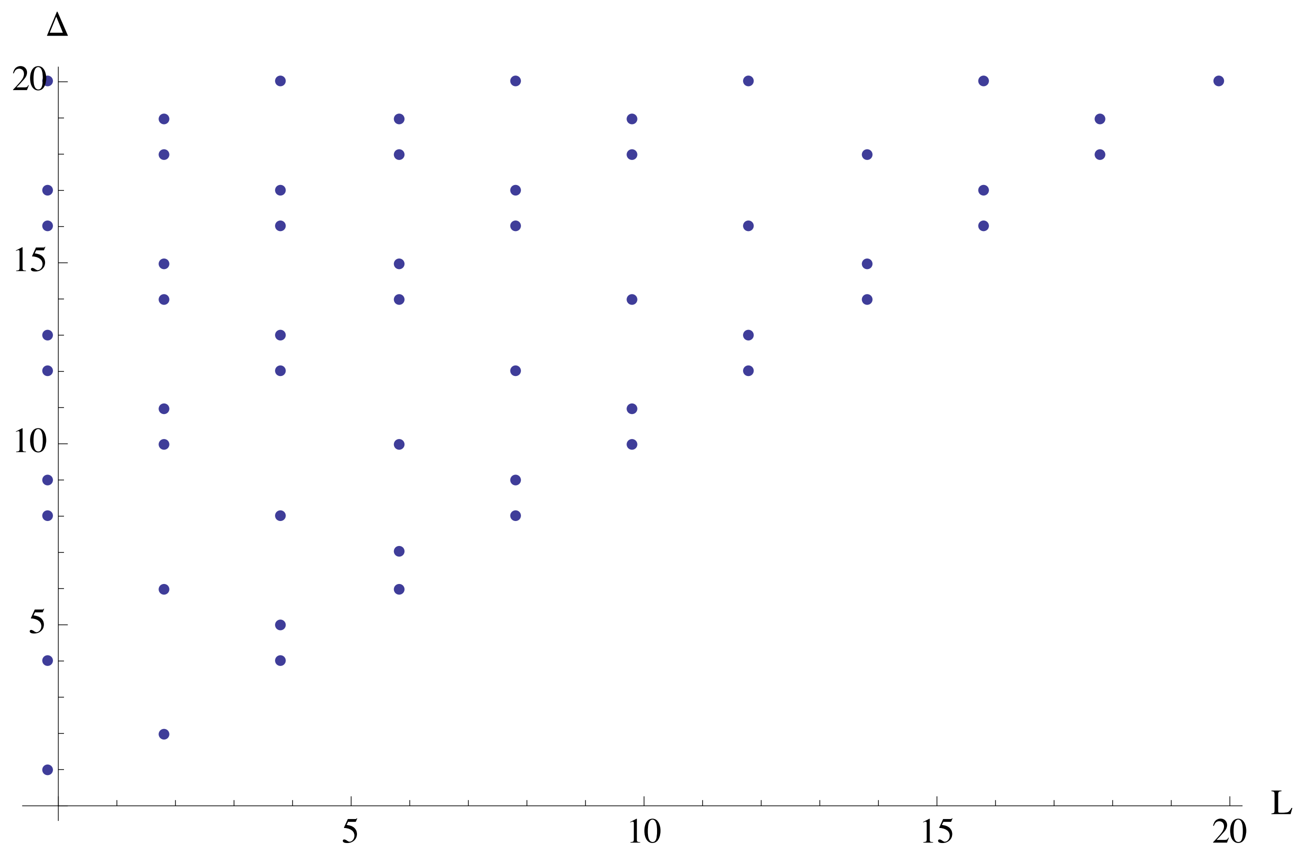
2. Look at several correlation functions simultaneously, e.g.

$$\langle \sigma\sigma\sigma\sigma \rangle \qquad \langle \epsilon\epsilon\sigma\sigma \rangle \qquad \langle \epsilon\epsilon\epsilon\epsilon \rangle$$

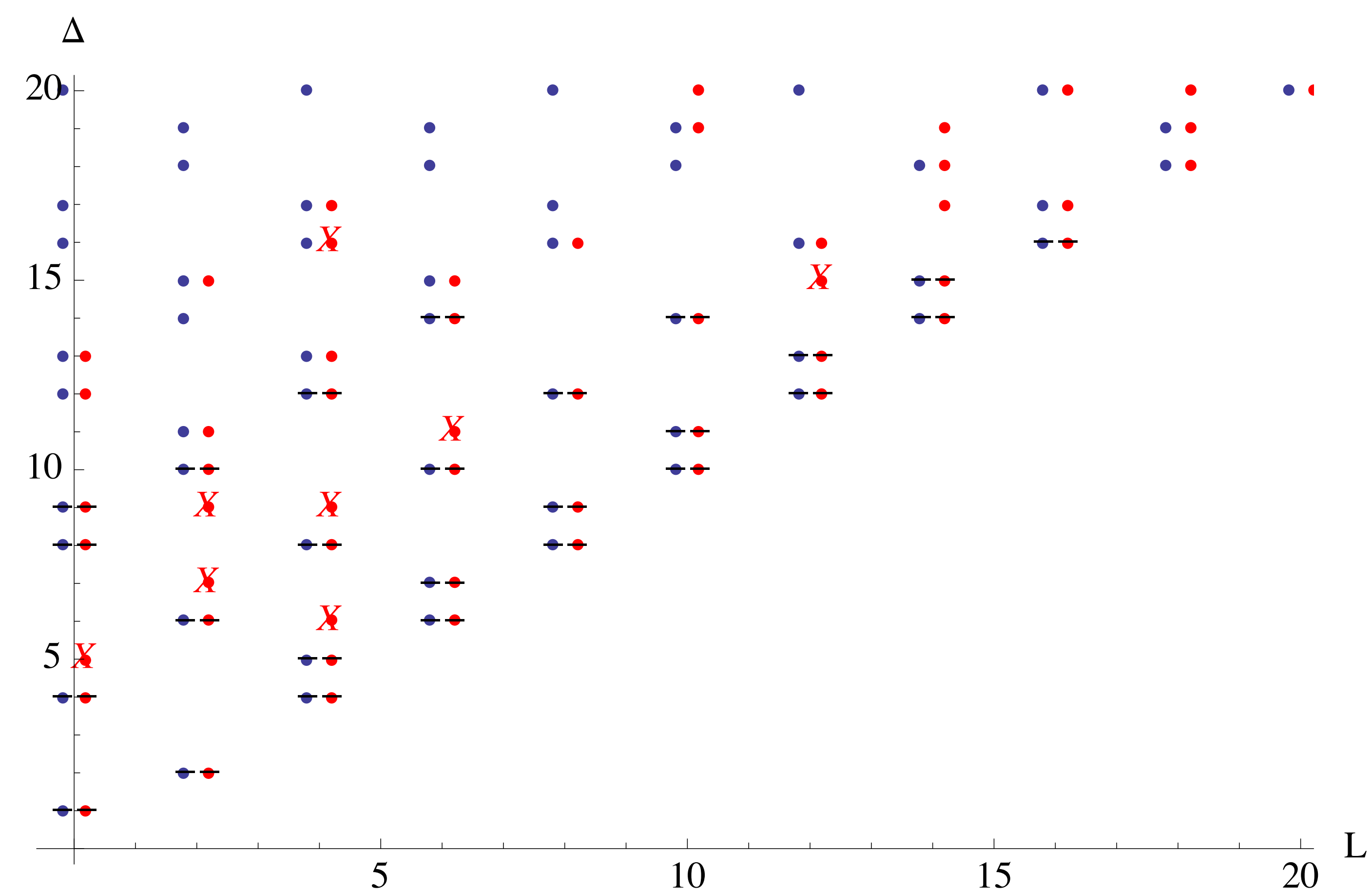
3. Full spectrum extraction at the boundary and the kinks



Exact 2D Ising spectrum:



Input exact Δ_σ and Δ_ε and allow all integer dimensions for others:



For 2D Ising done systematically by El-Showk & Paulos'2012

L	Δ_{EFM}	Δ	Err_{Δ} (%)	OPE_{EFM}	OPE	Err_{OPE} (%)	Err. Est. (%)
0	1.000003	1	0.00025	0.4999997	0.5	6.98E-05	1.1087E-05
	4.0003	4	0.0076	1.56241E-02	0.015625	0.0059	0.003
	8.0817	8	1.0	2.17003E-04	0.00021973	1.2	2.8
	30.2000	29	4.1	2.46649E-07	0.0017688	100.0	N/A
2	2.0000	2	0	1.76777E-01	0.176777	0.0001	0.00070
	5.9979	6	0.035	2.61754E-03	0.00262039	0.1	0.02
	7.8600	6	31	8.66110E-05	0.00262039	96.7	N/A
	10.6200	11	3.5	4.11441E-05	9.6505E-06	326.3	N/A
	14.3267	14	2.3	8.60258E-07	1.9167E-06	55.1	N/A
4	4.0000	4	0	2.09627E-02	0.0209631	0.0021	0.005
	5.0003	5	0.0063	5.52411E-03	0.00552427	0.0030	0.04
	7.9920	8	0.1	4.63914E-04	0.00046138	0.5	0.8
	11.4067	12	4.9	1.26831E-05	1.0886E-05	16.5	21.9
	15.2600	16	4.6	2.07807E-06	4.0479E-07	413.4	N/A
6	6.0000	6	0	3.69140E-03	0.00369106	0.0092	0.0006
	6.9978	7	0.031	1.23528E-03	0.00123526	0.0013	0.2
	10.0009	10	0.0089	9.15865E-05	9.1798E-05	0.2	2.3

Trying to do the same for 3D Ising (+Δ_σ determination using kinks)

[El-Showk, Paulos, Poland, Simmons-Duffin, S.R,,Vichi 'work in progress]

So far numerical approach was most successful in getting concrete results...

Can one get an analytic understanding of the resurrected bootstrap?

See e.g. [Fitzpatrick, Kaplan, Poland, Simmons-Duffin '12] [Komargodski, Zhiboedov'12] for analytic bootstrap results on large spin spectrum

If you want to learn more about CTFs in D>=3 and bootstrap: See recent lecture notes at my homepage.