Quantum Integrable Systems, Matrix models, and AGT correspondence

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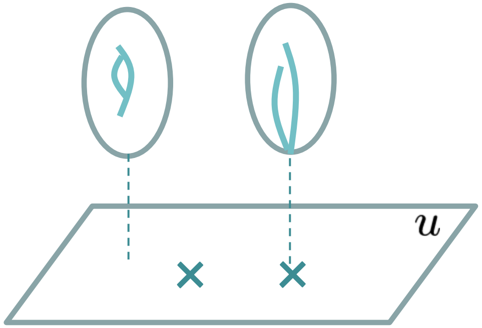
Introduction

N=2 supersymmetric gauge theory is very interesting framework where various results have been found.

In particular, the exact effective action can be determined by the Seiberg-Witten curve (in $(t,x) \in \mathbb{C} \times \mathbb{C}$) and differential

$$x^2 = \phi(t,u), \quad \lambda_{\mathsf{SW}} = x\,dt$$



Using these, the low energy prepotential is

$$a_i = \oint_{A_i} \lambda_{\mathsf{SW}}, \quad \frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} \lambda_{\mathsf{SW}}$$

Classical integrable systems

The relation between the Seiberg-Witten theory and the classical integrable system has been studied

[Gorsky et al., Donagi-Witten]
[Martinec-Warner, Itoyama-Morozov, ...]

Seiberg-Witten curve $\Longleftrightarrow$ spectral curve

$$x^2 = \phi(t, u)$$

meromorphic differential $\Longleftrightarrow$ Symplectic form

$$\Omega = d\lambda_{\mathrm{SW}} = \sum_i \frac{\partial x}{\partial u_i} du_i \wedge dt$$

$\Longrightarrow$ Commuting Hamiltonians: $\{u_i, u_j\} = 0$

M5-branes and Hitchin systems

[Donagi-Witten, Witten '97, Gaiotto '09]

In general, N=2, SU(2) gauge theories can be induced from 2 M5-branes wrapped on Riemann surfaces $\mathcal{C}_{g,n}$.

The Seiberg-Witten curve is written as

$$x^2 = \phi(t), \quad t; \text{coordinate on } \mathcal{C}_{g,n}$$

Associated integrable system is Hitchin system on $\mathcal{C}_{g,n}$ whose spectral curve is:

$$det(\Phi - x \cdot 1) = 0, \qquad \Phi \text{ is Higgs field}$$

Relation to quantum integrable systems

[Nekrasov-Shatashvili]

The Nekrasov partition function is

$$Z_{\mathsf{Nek}}(a, m_i; \epsilon_1, \epsilon_2) = \exp\left(-\frac{\mathcal{F}}{\epsilon_1 \epsilon_2} + \ldots\right)$$

The partition function is related to integrable system.

$$Z_{\mathsf{Nek}}(a, m_i; \epsilon_1, \epsilon_2) = \exp\left(-\frac{1}{\epsilon_1 \epsilon_2}(\mathcal{F}(\epsilon_1) + \mathcal{O}(\epsilon_2))\right)$$

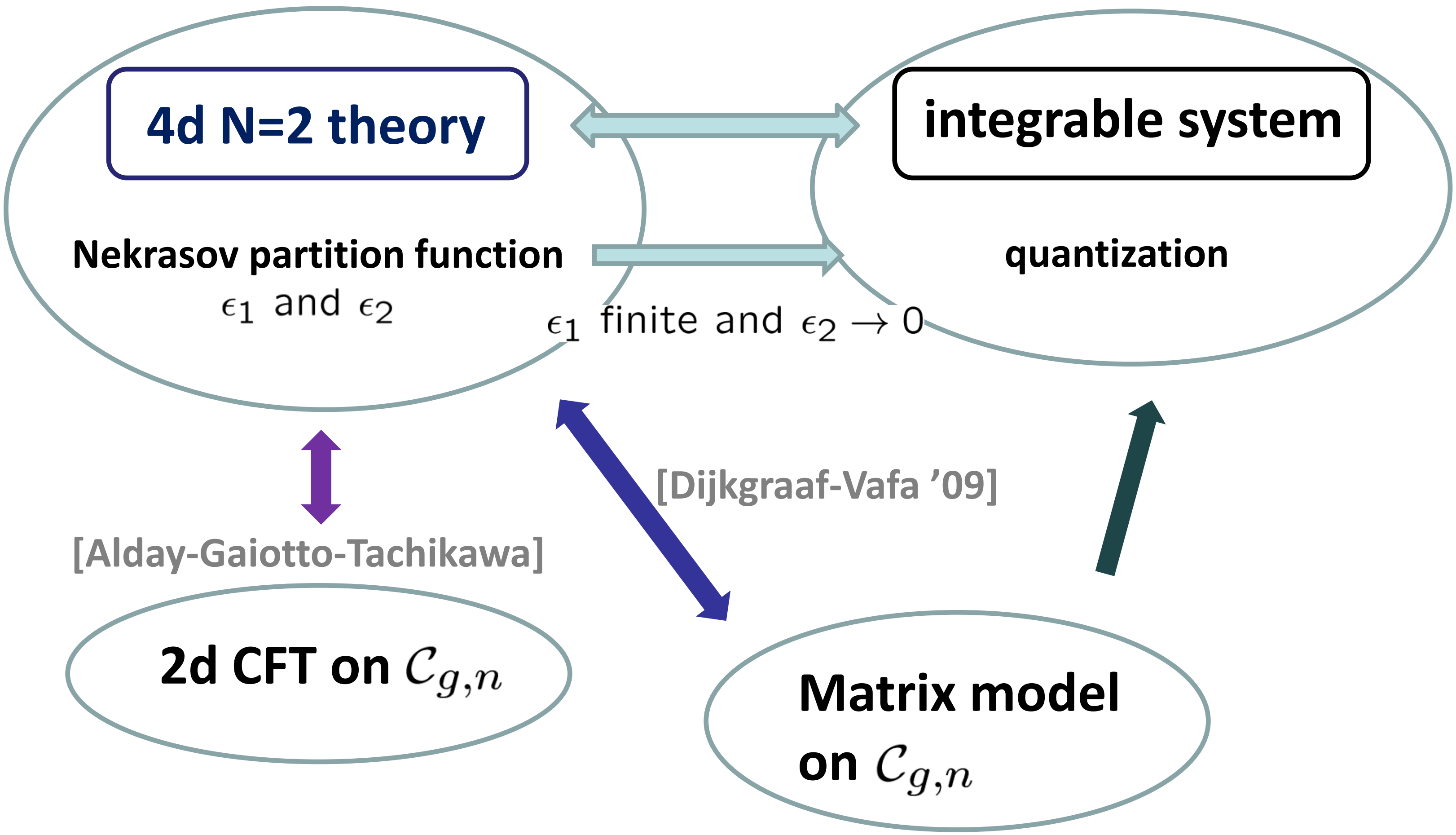
ϵ_1 finite and $\epsilon_2 \to 0$ **limit in Nekrasov partition function**

$\Longleftrightarrow$

Quantization of integrable system

(ϵ_1 *plays the role of the Planck constant*)

AGT and matrix models



Matrix models

[Dijkgraaf-Vafa ’09]

On a sphere:

$$Z^{\mathcal{C}_{0,n}} = \left(\prod_{I=1}^{N} \int d\lambda_I\right) \prod_{I<J} (\lambda_I - \lambda_J)^{-2b^2} \exp\left(\frac{b}{g_s} \sum_I W(\lambda_I)\right)$$

$$W(z) = \sum_{k=0}^{n-2} 2m_k \log(z - w_k),$$

On a torus:

$$Z^{\mathcal{C}_{1,n}} = \left(\prod_{I=1}^{N} \int d\lambda_I\right) \prod_{I<J} \vartheta_1(\lambda_I - \lambda_J)^{-2b^2} \exp\left(\frac{b}{g_s} \sum_I W(\lambda_I)\right)$$

$$W(z) = \sum_{k=1}^{n} 2m_k \log \vartheta_1(z - w_k) + 4\pi p z,$$

Loop equations

All useful information of the matrix model is included in the loop equation:

$$0 = -\langle R(z)^2 \rangle - (\epsilon_1 + \epsilon_2)\langle R(z)' \rangle + \langle R(z) \rangle W'(z) - f(z).$$

(for the sphere). We see that the loop equation in the $\epsilon_2 \to 0$ limit reduces to the differential equation of the corresponding Hitchin system

$$0 = -\epsilon_1^2 \frac{\partial^2}{\partial z^2}\Psi(z) + U(z)\Psi(z), \qquad \Psi(z) = \exp\left(\frac{1}{\epsilon_1}\int^z (\langle R(z') \rangle - W'(z')/2) dz'\right)$$

where

$$U(z) = \sum_{k=0}^{n-2} \frac{m_k(m_k + \epsilon_1)}{(z - w_k)^2} + \sum_k \frac{H_k}{z - w_k} - \sum_{k=0}^{n-2} \frac{c_k}{z - w_k}$$

Contents

1. *AGT correspondence and its extension*

AGT correspondence

[Alday-Gaiotto-Tachikawa]

M5-branes on $\mathbb{R}^4 \times \mathcal{C}_{g,n}$

$\mathcal{C}_{g,n}$: hypersurface in $(x,t) \in \mathbb{C} \times \mathbb{C}$

compactify on $\mathcal{C}_{g,n}$ [Gaiotto]

4d N=2 gauge theory $\longleftrightarrow$ **2d Liouville field theory on** $\mathcal{C}_{g,n}$

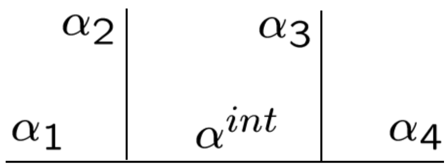
(Nekrasov instanton partition fuction) = (conformal block on $\mathcal{C}_{g,n}$)

Four-point conformal block

Let us consider the four-point correlation function which has the following form

$$\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle^{\text{full}}$$
$$= \int \frac{d\alpha^{int}}{2\pi} C(\alpha_1^*, \alpha_2, \alpha^{int}) C(\alpha^{int *}, \alpha_3, \alpha_4) |q^{\Delta_\alpha^{int} - \Delta_{\alpha_3} - \Delta_{\alpha_4}} \mathcal{B}(q)|^2$$

where *B* is expanded as $\mathcal{B}(q) = 1 + \mathcal{O}(q)$.



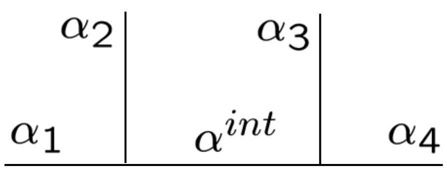
The conformal block *B* can also be written as

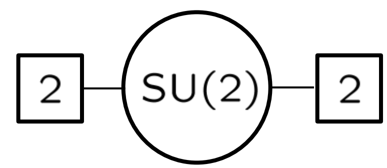
$$\begin{aligned}\mathcal{B}(q) &= \sum_{Y,W} q^{|Y|} \left\langle V_{\alpha_1} V_{\alpha_2} (L_{-Y} \phi_{\Delta^{int}}) \right\rangle Q^{-1}_{\Delta^{int}}(Y;W) \left\langle (L_{-W} \phi_{\Delta^{int}}) V_{\alpha_3} V_{\alpha_4} \right\rangle \\ Q_\Delta(Y;W) &= \langle L_{-Y} \phi_\Delta L_{-W} \phi_\Delta \rangle\end{aligned}$$

AGT relation (SU(2) theory with 4 flavors)

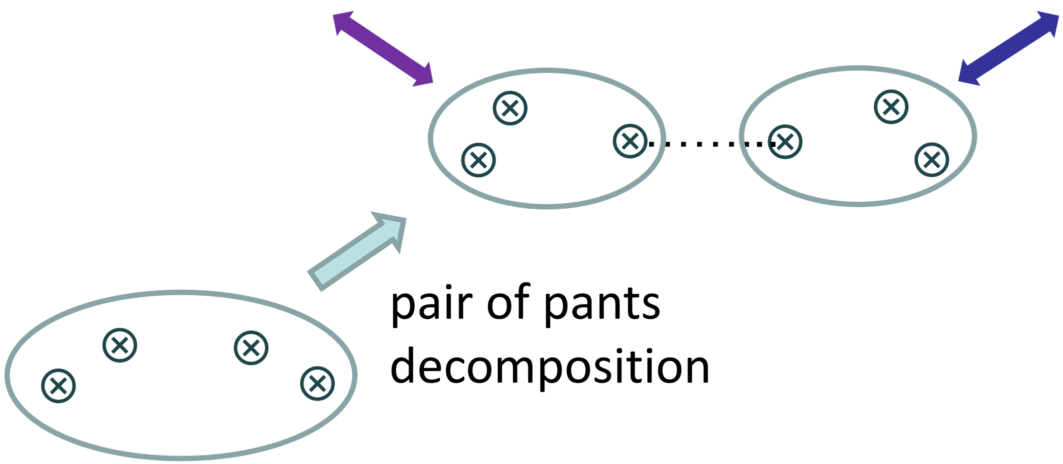
A simple example of the AGT relation is as follows:

$$\mathcal{B}(\alpha^{int}, \alpha_i, q; b) = Z_{\text{inst}}^{N_f=4}(a, m_i, q; \epsilon_1, \epsilon_2)$$



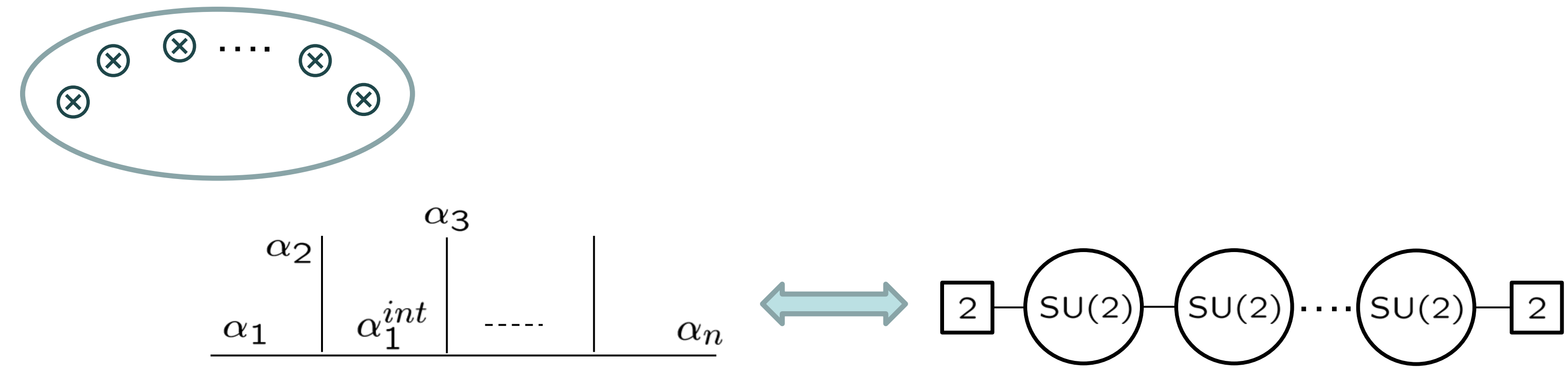


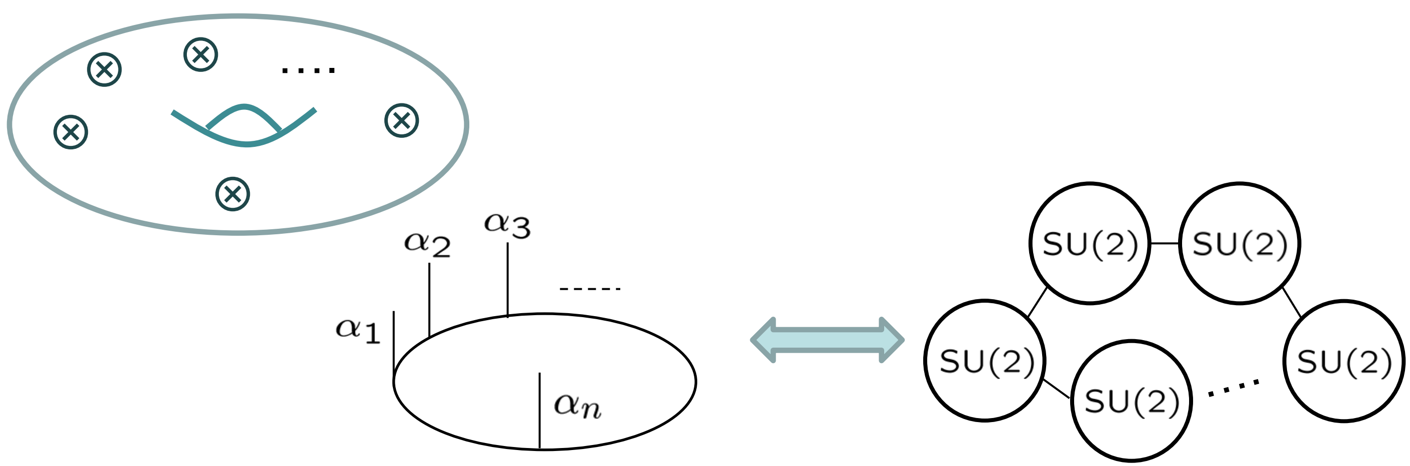
SU(2) w/ 4 flavors



pair of pants
decomposition

AGT relation (quiver gauge theories)





Identification of the parameters

External momenta = mass parameters

Internal momenta = vector multiplet scalar vevs

(complex structures) = (bare coupling constants)

deformation parameters : Liouville parameter $\epsilon_1 = \hbar b, \quad \epsilon_2 = \frac{\hbar}{b}.$

Seiberg-Witten curve is obtained by the insertion of the energy-momentum tensor:

$$x^2 = \phi(z) \equiv \frac{\langle T(z) \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle}{\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle},$$

and taking a limit where $\alpha^{int}, \alpha^i \gg b, 1/b$. $(\epsilon_{1,2} \to 0)$

Schrodinger equation via AGT

[Tschner, KM-Taki]

The differential equation which is satisfied by the conformal block with degenerate field due to the null condition:

$$(b^2 L_{-2} + (L_{-1})^2)\Phi_{2,1}(z) = 0$$

in the limit ϵ_1 finite and $\epsilon_2 \to 0$, can be identified with the Schrodinger equation

$$\mathcal{H}\Psi(z) = E\Psi(z), \quad \Psi(z) = \left\langle \Phi_{2,1}(z) \prod_{k=1}^{n} V_{m_k}(w_k) \right\rangle$$

(Ψ ***corresponds to the partition function in presence of a surface operator)*** [Alday-Gaiotto-Gukov-Tachikawa-Verlinde]

2. *Hitchin Hamiltonians from matrix models*

Beta-deformed matrix model

The matrix model corresponding to the conformal block on the sphere is

$$Z^{\mathcal{C}_{0,n}} = \left(\prod_{I=1}^{N} \int d\lambda_I \right) \prod_{I<J} (\lambda_I - \lambda_J)^{-2b^2} \exp\left(\frac{b}{g_s} \sum_I W(\lambda_I) \right)$$

$$W(z) = \sum_{k=0}^{n-2} 2m_k \log(z - w_k),$$

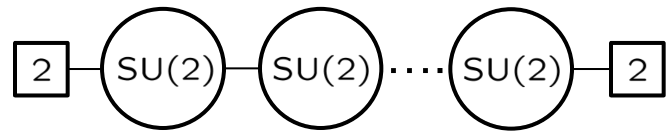
This is obtained from the free field representation:

$$\left\langle \left(\int d\lambda_I : e^{2b\phi(\lambda_I)} : \right)^N \prod_{k=0}^{n-1} V_{m_k}(w_k) \right\rangle_{\text{free on } \mathcal{C}_0}$$

with the OPE: $\phi(z)\phi(\omega) \sim -\frac{1}{2}\log(z-\omega)$ *and the momentum conservation* $\sum m_k + m_\infty - bg_s N + g_s Q = 0.$

Matrix model approach to AGT

Matrix model $Z^{\mathcal{C}_{0,n}}$ $\Rightarrow$



- in the large N limit, the *spectral curve agrees with the Seiberg-Witten curve* **[Dijkgraaf-Vafa, Eguchi-KM]**

  the Coulomb moduli corresponds to the filling fractions bg_sN_i

- the large N free energy agrees with the prepotential **[Eguchi-KM]**

$$Z^{\mathcal{C}_{0,4}} = e^{F_m/g_s^2}, \qquad F_m = \underbrace{F_0}_{=\,\mathcal{F}_{pre}} + \mathcal{O}(g_s)$$

Loop equation of Beta-deformed matrix model

We define the resolvent

$$R(z) = bg_s \sum_I \frac{1}{z - \lambda_I}$$

The resolvent satisfies the loop equation:

$$0 = -\langle R(z)^2 \rangle - (\epsilon_1 + \epsilon_2)\langle R(z)' \rangle + \langle R(z) \rangle W'(z) - f(z).$$

where

$$\begin{aligned} f(z) &= bg_s \left\langle \sum_I \frac{W'(z) - W'(\lambda_I)}{z - \lambda_I} \right\rangle = \sum_{k=0}^{n-2} \frac{c_k}{z - w_k}, \\ c_k &= -bg_s \left\langle \sum_I \frac{2m_k}{\lambda_I - w_k} \right\rangle = \frac{\partial F_m}{\partial w_k}. \end{aligned}$$

Large N limit and spectral curve

First of all, we consider the limit where $N \to \infty$ and $g_s \to 0$.

In this case $\langle R(z)^2 \rangle \to \langle R(z) \rangle^2$ and $\epsilon_{1,2} \to 0$, thus we obtain

$$0 = -\langle R(z) \rangle^2 + \langle R(z) \rangle W'(z) - f(z).$$

Or, by defining $x = \langle R(z) \rangle - \frac{W'(z)}{2}$

$$x^2 = \sum_{k=0}^{n-2} \frac{m_k^2}{(z-w_k)^2} + \sum_k \frac{H_k - c_k}{z - w_k}$$

This is the same as the Seiberg-Witten curve of the quiver gauge theory.

The $\epsilon_2 \to 0$ limit

$b \to \infty$ and $g_s \to 0$ with bg_s fixed.

In this limit, the loop equation reduces to

$$0 = -x^2 - \epsilon_1 x' + U(z), \quad \text{or} \quad 0 = -\epsilon_1^2 \frac{\partial^2}{\partial z^2}\Psi(z) + U(z)\Psi(z),$$

with the *wave-function*

$$\Psi(z) = \exp\left(\frac{1}{\epsilon_1}\int^z x(z')dz'\right)$$

where

$$U(z) = \sum_{k=0}^{n-2} \frac{m_k(m_k+\epsilon_1)}{(z-w_k)^2} + \sum_k \frac{H_k}{z-w_k} - \sum_{k=0}^{n-2} \frac{c_k}{z-w_k}$$

The H_k are exactly same as the Hamiltonians and c_k are their eigenvalues:

$$H_k = \sum_{\ell(\neq k)} \frac{2m_k m_\ell}{w_k - w_\ell}$$

Wave-function and resolvent

cf) [Marshakov-Mironov-Morozov]

The integral representation of the (n+1)-point conformal block with the degenerate field insertion can be obtained from

$$Z_{deg}^{\mathcal{C}_{0,n+1}} = \left\langle e^{-\frac{\phi(z)}{b}} \left(\int d\lambda e^{2b\phi(\lambda)} \right)^N \prod_{k=0}^{n-1} V_{\frac{m_k}{g_s}}(w_k) \right\rangle$$

By using this, we obtain, in the $\epsilon_2 \to 0$ limit,

$$\frac{Z_{deg}^{\mathcal{C}_{0,n+1}}}{Z^{\mathcal{C}_{0,n}}} \to \exp\left(\frac{1}{\epsilon_1} \int^z x(z')dz' \right) = \Psi(z)$$

Thus, we have seen that the wave-function corresponds to the conformal block with the degenerate field.

Generalized matrix model

[Dijkgraaf-Vafa, KM-Yagi]

The matrix model corresponding to the conformal block on the torus is

$$Z^{\mathcal{C}_{1,n}} = \left(\prod_{I=1}^{N} \int d\lambda_I \right) \prod_{I<J} \vartheta_1(\lambda_I - \lambda_J)^{-2b^2} \exp\left(\frac{b}{g_s} \sum_I W(\lambda_I) \right)$$

$$W(z) = \sum_{k=1}^{n} 2m_k \log \vartheta_1(z - w_k) + 4\pi p z,$$

This can be obtained from the Liouville correlator and the free field representation by using the method of [Goulian-Li]

The filling fraction $bg_s N_i$ and p correspond to the internal momenta (and the Coulomb moduli).

Wave-function and resolvent

Similar argument shows that the conformal block on the torus with the degenerate field can be written as

$$\frac{Z_{deg}^{\mathcal{C}_{1,n+1}}}{Z^{\mathcal{C}_{1,n}}} \to \exp\left(\frac{1}{\epsilon_1}\int^z x(z')dz'\right) = \Psi(z)$$

where

$$x = \langle R(z)\rangle - \frac{W'(z)}{2}, \qquad R(z) = bg_s \sum_I \frac{\vartheta_1'(z-\lambda_I)}{\vartheta_1(z-\lambda_I)}$$

R is the analog of the resolvent in the matrix model.

Loop equation

The loop equation in the $\epsilon_2 \to 0$ limit is

$$
\begin{aligned}
0 &= -x^2 - \epsilon_1 x' + \sum_k m_k(m_k + \epsilon_1)\mathcal{P}(z - w_k) \\
&\quad + \sum_k \frac{\vartheta_1'(z - w_k)}{\vartheta_1(z - w_k)}\left(H_k - \frac{\partial F^{\mathcal{C}_{1,n}}}{\partial w_k}\right) + H_0 + 4\frac{\partial F^{\mathcal{C}_{1,n}}}{\partial \ln q},
\end{aligned}
$$

where

$$
\begin{aligned}
H_k &= 4\pi p m_k + 2\sum_{\ell(\neq k)} m_k m_\ell \frac{\vartheta_1'(w_k - w_\ell)}{\vartheta_1(w_k - w_\ell)}, \\
H_0 &= 4\pi^2 p^2 - \eta_1 \sum_k m_k(m_k + \epsilon_1) + \frac{1}{2}\sum_{k\neq\ell} m_k m_\ell \frac{\vartheta_1''(w_k - w_\ell)}{\vartheta_1(w_k - w_\ell)}.
\end{aligned}
$$

These will agree with the Hamiltonians of the Hitchin system.

One-punctured torus

The equation is simplified in the one puncture case:

$$0 = -x(z)^2 - \epsilon_1 x'(z) + m_1(m_1+\epsilon_1)\mathcal{P}(z) - 4u(\epsilon_1),$$

where

$$u(\epsilon_1) = -\pi^2 p^2 + \frac{\partial}{\partial \ln q}\left(\mathcal{F}^{\mathcal{C}_{1,1}} - 2m_1(m_1+\epsilon_1)\ln\eta\right),$$

In terms of the wave-function, we obtain

$$\left[-\epsilon_1^2\frac{\partial^2}{\partial z^2} + m_1(m_1+\epsilon_1)\mathcal{P}(z)\right]\Psi = 4u(\epsilon_1)\Psi.$$

The left hand side is the Calogero-Moser Hamiltonian and u is the eigenvalue.

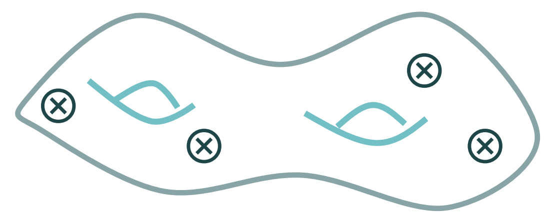
3. Hitchin system

Gauge theories and Hitchin systems

[Donagi-Witten]

The gauge theories induced from the M5-branes are related with the Hitchin systems on $\mathcal{C}_{g,n}$, whose phase space is $T^*\mathcal{A}$ where $\mathcal{A}$ is moduli space of G-bundle over $\mathcal{C}_{g,n}$.

This is specified by $(A,\ \Phi)$ which satisfy



$$\bar{\partial}\Phi = \sum_{k=1}^{n} \nu_k \delta(z - w_k)$$

We consider SL(2,C) bundle corresponding to SU(2) gauge theories. The Hamiltonians can be read from

$$\mathrm{Tr}\Phi^2 = \sum_k \mathrm{Tr}\nu_k^2 \eta_k(z) + H_i \zeta_i(z)$$

On a sphere

On a punctured sphere $\mathcal{C}_{0,n}$, the Higgs field reads

$$\Phi = \sum_k \frac{\nu_k}{z - w_k} \frac{dz}{2\pi i}$$

Thus, the quadratic differential is

$$\mathrm{Tr}\Phi^2 = \sum_k \left(\frac{J_k^2}{(z - w_k)^2} + \frac{H_k}{z - w_k} \right) \left(\frac{dz}{2\pi i} \right)^2$$

where

$$J_k^2 = \mathrm{Tr}\nu_k^2, \quad H_k = 2 \sum_{l \neq k} \frac{1}{w_k - w_l} \mathrm{Tr}\nu_l \nu_k$$

On a torus

On a punctured sphere $\mathcal{C}_{1,n}$, the Higgs field reads

$$\Phi = \left(\sum_k \nu_k \frac{\vartheta_1'(z-w_k)}{\vartheta_1(z-w_k)} + 2\pi i p \right) \frac{dz}{2\pi i}$$

Thus, the quadratic differential is

$$\mathrm{Tr}\Phi^2 = \sum_k \left(\mathcal{P}(z-w_k) J_k^2 + \frac{\vartheta_1'(z-w_k)}{\vartheta_1(z-w_k)} H_k + H_0 \right) \left(\frac{dz}{2\pi i} \right)^2,$$

where

$$\begin{aligned} H_k &= 2 \sum_{l \neq k} \mathrm{Tr}\nu_k \nu_l \frac{\vartheta_1'(w_k-w_l)}{\vartheta_1(w_k-w_l)} + 4\pi i \mathrm{Tr}\nu_k p, \\ H_0 &= -4\pi^2 \mathrm{Tr}p^2 - \eta_1 \sum_k \mathbf{J}_k^2 + \frac{1}{2} \sum_{k,l;k\neq l} \mathrm{Tr}\nu_k \nu_l \frac{\vartheta_1''(w_k-w_l)}{\vartheta_1(w_k-w_l)}. \end{aligned}$$

4. Conclusion and discussion

We proposed that the generalized matrix model can be used to see the relation between the gauge theory and the integrable system, by making use of the AGT correspondence.

We derived the differential equation from the loop equation of the matrix model associated with the conformal block with the degenerate field.

- Generalized matrix models on higher genus Riemann surfaces **[Bonelli-KM-Tanzini-Yagi]**
- SU(N)/Toda/multi-matrix generalization

Thank you very much for your attention!